BCS-like action and Lagrangian from the gradient expansion of the determinant of Fermi fields in QCD-type, non-Abelian gauge theories with chiral anomalies

(Derivation for an effective action of BCS quark pairs with the Hopf invariant $\Pi_3(S^2) = \mathsf{Z}$ as a nontrivial topology.)

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Abstract

An effective field theory of BCS quark pairs is derived from an ordinary QCD-type path integral with $\mathrm{SU}_c(N_c = 3)$ non-Abelian gauge fields. We consider the BCS quark pairs as constituents of nuclei and as the remaining degrees of freedom in a coset decomposition $\mathrm{SO}(N_0, N_0) \,/\, \mathrm{U}(N_0) \otimes \mathrm{U}(N_0)$ of a corresponding total self-energy matrix taking values as generator within the $\mathrm{so}(N_0, N_0)$ Lie algebra. The underlying dimension $(N_0 = N_f \cdot 4_{\hat{\gamma}} \cdot N_c)$ is determined by the product of isospin- '$N_f = 2$' (flavour- '$N_f = 3$') degrees of freedom, by the 4×4 Dirac gamma matrices with factor '$4_{\hat{\gamma}}$' and the colour degrees of freedom '$N_c = 3$'; therefore, the smallest, total self-energy generator has Lie algebra $\mathrm{so}(N_0, N_0)$ with $N_0 = 24$. We distinguish between a total unitary sub-symmetry $\mathrm{U}(N_0)$ for purely density related parts of the quarks, which are taken into account as background fields and as invariant vacuum states in a SSB, and between the BCS terms of quarks as coset elements $\mathrm{so}(N_0, N_0) \,/\, \mathrm{u}(N_0)$. The self-energies are obtained by dyadic products of anomalous doubled Fermi fields and subsequent HST's where we only use the reproducing property of Gaussian factors in Fourier transformations. These HST's are sufficient to achieve a path integral entirely determined by self-energy matrices for the coset decomposition. Finally, we can compare the derived effective actions of BCS quark pairs with the effective Skyrme Lagrangian, which is classified by the homotopy group $\Pi_3(\mathrm{SU}(2)\,) = \mathsf{Z}$ for topological solitons as the baryons, and attain the astonishing result that our derived effective actions of BCS quark pairs are more closely related to the Skyrme-Faddeev field theory with the nontrivial Hopf mapping $\Pi_3(S^2) = \mathsf{Z}$.

Keywords : gradient expansion of determinants, chiral anomalies, nontrivial topology, Hopf mapping, Hopf invariant, spontaneous chiral symmetry breaking, effective Skyrme-Lagrangian

PACS : **12.39.Dc , 12.39.Fe , 11.30.Rd , 11.10.Ef**

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1 Introduction

1.1 Symmetries and dimensions of self-energies for BCS quark pairs and densities

The strong interaction of hadrons in nuclei is described by very different concepts which range from semi-empirical mass formulas [1]-[4], nuclear shell models [5], effective Lagrangians of mesons and baryons [6] to QCD- or even string-theory [7, 8]. One common part of these models and theories concerns the strong spin-pairing force of nucleons so that one has already to include spin pairing corrections in the original Bethe-Weizäcker mass formula. This extraordinary pairing force of protons and neutrons has even provided own models or theories as the 'interacting boson approximation' or the notion of 'nuclear superfluidity' of spin paired fermionic hadrons [9, 10].

In this paper we consider BCS quark pairs as constituents of nuclei and as the remaining degrees of freedom in a coset decomposition of an ordinary (but subsequently transformed) QCD path integral with axial gauge fixing. We follow the given coset decomposition $\mathrm{Osp}(S,S|2L)\,/\,\mathrm{U}(L|S)\otimes\mathrm{U}(L|S)$ of the ortho-symplectic super-group $\mathrm{Osp}(S,S|2L)$ for bosonic and fermionic atoms in a trap potential according to Refs. [11, 12] and 'strictly' derive a transformed path integral with a total self-energy matrix taking (even)-values in the $\mathrm{so}(N_0,N_0)$ Lie algebra. The dimension N_0, (respectively the anomalous doubled case with BCS quark pairs and overall dimension $(N_0\,,\,N_0)=2\,N_0$), is determined by the product of $N_f=2$ isospin- (or $N_f=3$ flavour-) degrees of freedom, the $4_{\hat{\gamma}}\times 4_{\hat{\gamma}}$ Dirac gamma matrices in 3+1 dimensional spacetime and the $\mathrm{SU}_c(N_c=3)$ gauge field degrees of freedom of QCD. This yields in total a unitary sub-symmetry $\mathrm{U}(N_0)$ for purely density related parts of the quarks with $N_0=N_f\times 4_{\hat{\gamma}}\times N_c$. The self-energy sub-matrix for a single block density section of quarks therefore has the dimension $N_0=(N_f=2)\times 4_{\hat{\gamma}}\times(N_{c=3}=3)=24$ or with strangeness $(N_f=3)$, $N_0=(N_f=3)\times 4_{\hat{\gamma}}\times(N_{c=3}=3)=36$. The anomalous doubling of the fermionic quark fields then leads to the total $\mathrm{SO}(N_0,N_0)$ symmetry for the self-energy which comprises in the block diagonals the density related $\mathrm{U}(N_0)$ symmetry and in the off-diagonals blocks the anti-symmetric sub-matrices for BCS quark pairs; the latter complex, even-valued BCS parameter fields originate from the coset decomposition $\mathrm{SO}(N_0,N_0)\,/\,\mathrm{U}(N_0)\otimes\mathrm{U}(N_0)$. In advance we symbolically indicate the density (or subgroup part) and BCS terms of the anomalous doubling of fermionic quark fields in relation (1.1); this equation also denotes the scheme of a spontaneous symmetry breaking (SSB) with the coset decomposition $\mathrm{SO}(N_0,N_0)\,/\,\mathrm{U}(N_0)\otimes\mathrm{U}(N_0)$ and the invariant vacuum or ground states of the density related unitary symmetry $\mathrm{U}(N_0)$

$$\begin{pmatrix}\text{total so}(N_0,N_0)\text{ self-energy}\\ 2\,N_0\cdot(2\,N_0-1)/2\quad\text{real parameters}\end{pmatrix}^{ab}= \tag{1.1}$$

$$=\begin{pmatrix}\begin{pmatrix}\mathrm{u}(N_0)\text{ density '11'}\\ (\,N_0\,)^2\text{ real parameters}\end{pmatrix}^{11} & \begin{pmatrix}\imath\times\text{(anti-symmetric BCS pairs) '12'}\\ N_0\cdot(N_0-1)/2\text{ complex parameters}\end{pmatrix}^{12}\\ \begin{pmatrix}\imath\times\text{(anti-symmetric BCS pairs)}^\dagger\text{ '21'}\\ N_0\cdot(N_0-1)/2\text{ complex parameters}\end{pmatrix}^{21} & \begin{pmatrix}\mathrm{u}^T(N_0)\text{ transposed density '22' of '11' block}\\ \textit{same }(\,N_0\,)^2\text{ real parameters of '11' part}\end{pmatrix}^{22}\end{pmatrix}^{ab}.$$

This total self-energy of $\mathrm{so}(N_0,N_0)$ with BCS pairing in the off-diagonal blocks '12' and '21' is achieved by various Hubbard Stratonovich transformations (HST) from the original, ordinary QCD path integral with axial gauge fixing. We double the original, odd-valued Fermi- or quark fields

$$\psi_M^\dagger(x_p)\;\psi_M(x_p)\rightarrow\frac{1}{2}\Big(\psi_M^\dagger(x_p)\;\psi_M(x_p)-\psi_M^T(x_p)\;\psi_M^*(x_p)\Big)\;, \tag{1.2}$$

and introduce dyadic products of these which then specify the even-valued, hermitian self-energies. Although the various HST's to self-energies involve intricate manipulations (section 3), we need only to apply the reproducing property of Gaussian integrals in a Fourier transformation; the Gaussian factor of quartic Fermi fields (or of quadratic multiples of their dyadic products in a trace relation) is equivalent to the Gaussian integral of the self-energy matrix with linear coupling to bilinear Fermi fields or to their corresponding dyadic product in a trace relation. This kind of HST with Gaussian factors and integrals is also applicable for the eight gauge fields $A_{\alpha;\mu}(x_p)$ of $\mathrm{SU}_c(N_c=3)$, $(\alpha=1,\ldots,8\;,\;\mu=0,1,2,3)$ [2]. Although there occur three- and four-point vertices of these gauge fields, we only need the reproducing property of

[2]The semicolon ';' of $A_{\alpha;\mu}(x_p)$ in this paper just separates the internal indices $\alpha,\,\beta,\,\gamma=1,\ldots,8$ from the Lorentz-indices $\kappa,\,\lambda,\,\mu,\,\nu=0,\ldots,3$ of 3+1 dimensional spacetime, but does not denote something like a covariant derivative.

Gaussian integrals in a Fourier transformation in order to attain an anti-symmetric self-energy matrix for the quadratic gauge field strength tensor $\hat{F}_{\alpha}^{\mu\nu}(x_p)$. The Gaussian factor of this field strength tensor of gauge fields is transformed by a Gaussian integral of the corresponding anti-symmetric self-energy matrix with remaining linear coupling. Since the gauge field strength tensor $\hat{F}_{\alpha}^{\mu\nu}(x_p)$ has only quadratic terms of the original eight gauge fields $A_{\alpha;\mu}(x_p)$ $(\alpha = 1,\ldots,8)$, we can integrate over the remaining quadratic gauge fields in the linear coupling between the gauge field strength tensor and corresponding self-energy. This unfinished, quadratic integral of $A_{\alpha;\mu}(x_p)$ is modified by the the 'minimal' coupling principle of a single gauge field to bilinear quark fields; nevertheless, the remaining Gaussian integral of gauge fields can be performed and results into the well-known self-interaction of the self-energy matrix for the gauge field strength tensor. Similar HST's are applied for the quartic interactions of odd-valued quark fields which are finally kept only in a bilinear, anomalous doubled kind $(\psi^*\,,\,\psi)\,(\hat{H}\,,\,-\hat{H}^T)\,(\psi\,,\,\psi^*)^T$ for the doubled one-particle operator $(\hat{H},\,-\hat{H}^T)$; thus they can be removed by odd-valued, anomalous doubled Gaussian integration properties of Fermi fields.

However, the resulting Fermi determinant contains the total self-energy matrix so(N_0,N_0) and the 'colour' dressed quark density parts, combined to a composed gauge field $\mathcal{V}_{\alpha;\mu}(x_p)$ in place of the original gauge field $A_{\alpha;\mu}(x_p)$, so that further simplification is not directly obvious. According to Derrick's theorem [13], an effective Lagrangian should have up to order of 'four' derivative terms in order to allow for stable, static energy configurations in 3+1 spacetime dimensions; the second and fourth order gradient parts prevent a scaling of the particular configuration to arbitrary small or large sizes in the three dimensional coordinate space integrations over the static Hamiltonian density. The intensive, but straightforward HST's in section 3, substitute the original QCD path integral (2.25-2.27)

$$Z[\hat{\mathfrak{J}},J_{\psi},\hat{J}_{\psi\psi},\hat{\mathfrak{j}}^{(\hat{F})}] = \int d[\psi_M^{\dagger}(x_p)\,,\,\psi_M(x_p)]\;\;d[A_{\alpha;\mu}(x_p)]\left\{\prod_{\{x_p,\alpha\}}^{\substack{n^{\mu}=(0,\vec{n})\\ \vec{n}\cdot\vec{n}=1}}\delta\big(n^{\mu}\,A_{\alpha;\mu}(x_p)\big)\right\}\times \tag{1.3}$$
$$\times\;\exp\Big\{\imath\,\mathcal{A}[\psi,\hat{D}_{\mu}\psi,\hat{F}]-\imath\,\mathcal{A}_S[\hat{\mathfrak{J}},J_{\psi},\hat{J}_{\psi\psi},\hat{\mathfrak{j}}^{(\hat{F})}]\Big\}\;;$$

$$\mathcal{A}[\psi,\hat{D}_{\mu}\psi,\hat{F}] = \int_C d^4x_p\bigg\{-\frac{1}{4}\,\hat{F}_{\alpha;\mu\nu}(x_p)\,\hat{F}_{\alpha}^{\mu\nu}(x_p)+ \tag{1.4}$$
$$-\;\psi_N^{\dagger}(x_p)\Big[\hat{\beta}\Big(\hat{\gamma}^{\mu}\hat{\partial}_{p,\mu}-\imath\,\hat{\varepsilon}_p-\imath\,\hat{\gamma}^{\mu}\,A_{\alpha;\mu}(x_p)\,\hat{t}_{\alpha}+\hat{m}\Big)\Big]_{N;M}\psi_M(x_p)\bigg\}\;;$$

$$\hat{\varepsilon}_p = \hat{\beta}\,\eta_p\,\varepsilon_+\;;\qquad \varepsilon_+>0\,, \tag{1.5}$$

by the path integral relations (3.110-3.116)

$$\hat{\mathfrak{O}}_{N;M}^{ba}(y_q,x_p) = \bigg\{\hat{\mathcal{H}}+\Big(\hat{T}^{-1}\hat{\mathcal{H}}\hat{T}-\hat{\mathcal{H}}\Big)+\hat{T}^{-1}\,\hat{I}\,\hat{S}\,\eta_q\frac{\hat{\mathfrak{J}}_{N';M'}^{b'a'}(y_q,x_p)}{\mathcal{N}}\eta_p\,\hat{S}\,\hat{I}\,\hat{T}\bigg\}_{N;M}^{ba}(y_q,x_p)\;; \tag{1.6}$$

$$\mathcal{A}_{DET}[\hat{T},\hat{\mathcal{V}};\hat{\mathfrak{J}}] = \frac{1}{2}\mathop{\mathrm{TR}}_{\int_C d^4x_p\,\eta_p}^{a(=1,2)}\mathop{\mathfrak{tr}}_{N_f,\hat{\gamma}_{mn}^{(\mu)},N_c}\bigg(\ln\Big[\hat{\mathfrak{O}}_{N;M}^{ba}(y_q,x_p)\Big]-\ln\Big[\hat{\mathcal{H}}_{N;M}^{ba}(y_q,x_p)\Big]\bigg)\;; \tag{1.7}$$

$$\mathcal{A}_{J_{\psi}}[\hat{T},\hat{\mathcal{V}};\hat{\mathfrak{J}}] = \frac{1}{2}\int_C d^4x_p\,d^4y_q\;\times \tag{1.8}$$
$$\times\,J_{\psi;N}^{\dagger,b}(y_q)\,\hat{I}\Big(\hat{T}(y_q)\,\hat{\mathfrak{O}}_{N';M'}^{-1;b'a'}(y_q,x_p)\,\hat{T}^{-1}(x_p)-\hat{\mathcal{H}}_{N;M}^{-1;ba}(y_q,x_p)\Big)\,\hat{I}\,J_{\psi;M}^{a}(x_p)\;;$$

$$Z[\hat{\mathfrak{J}},J_{\psi},\hat{J}_{\psi\psi},\hat{\mathfrak{j}}^{(\hat{F})}] = \bigg\langle \boldsymbol{Z\Big[\hat{\mathcal{V}}(x_p);\hat{\mathfrak{S}}^{(\hat{F})},s_{\alpha},\hat{\mathfrak{B}}_{\hat{F}},\hat{\mathfrak{b}}^{(\hat{F})},\hat{\mathfrak{U}}_{\hat{F}},\hat{\mathfrak{v}}_{\hat{F}};\sigma_D;\hat{\mathfrak{j}}^{(\hat{F})};\mathbf{Eq.\;(3.59)}\Big]}\times \tag{1.9}$$
$$\times\int d[\hat{T}^{-1}(x_p)\,d\hat{T}(x_p)]\;\;Z_{\hat{J}_{\psi\psi}}[\hat{T}]\;\exp\Big\{\mathcal{A}_{DET}[\hat{T},\hat{\mathcal{V}};\hat{\mathfrak{J}}]\Big\}\;\exp\Big\{\imath\,\mathcal{A}_{J_{\psi}}[\hat{T},\hat{\mathcal{V}};\hat{\mathfrak{J}}]\Big\}\bigg\rangle\,.$$

Despite intricate appearance of the transformed path integral (1.9,3.116), we can attain through subsequent HST's and a coset decomposition $\mathrm{SO}(N_0,N_0)\,/\,\mathrm{U}(N_0)\otimes\mathrm{U}(N_0)$ a clear separation into BCS terms with coset matrices $\hat{T}(x_p) =$

$\exp\{-\hat{Y}(x_p)\}$ (last line of (1.9,3.116))

$$\hat{T}^{ab}_{M;N}(x_p) = \Big(\exp\Big\{-\hat{Y}^{a'b'}_{M';N'}(x_p)\Big\}\Big)^{ab}_{M;N} ; \tag{1.10}$$

$$\hat{Y}^{ab}_{M;N}(x_p) = \begin{pmatrix} 0 & \hat{X}_{M;N}(x_p) \\ \hat{X}^{\dagger}_{M;N}(x_p) & 0 \end{pmatrix}^{a\neq b} ; \quad \hat{X}^T_{M;N}(x_p) = -\hat{X}_{M;N}(x_p) , \tag{1.11}$$

and a quark density part with various transformed gauge field parts (first line of (1.9,3.116) in boldface symbols with path integral (3.59)). The original gauge field $A_{\alpha;\mu}(x_p)$ in (1.3-1.5,2.25-2.27) is exchanged in (1.9,3.116) by the gauge field $\mathcal{V}_{\alpha;\mu}(x_p)$ (3.60) which is composed of colour dressed quark densities and various gauge field combinations with auxiliary fields for axial gauge fixing. This replacement $\mathcal{V}_{\alpha;\mu}(x_p)$ (3.60) of the original gauge field $A_{\alpha;\mu}(x_p)$ is contained in both parts of the final transformed path integral (1.9,3.116), the density and gauge field related, first part (3.59) and the coset part with $\hat{T}(x_p)$ and anti-symmetric BCS generator $\hat{X}(x_p)$, $\hat{X}^{\dagger}(x_p)$. The path integral (1.9,3.116), which is equivalent to the original one (1.3-1.5,2.25-2.27), allows for the separate approximation of the density part (3.59) which yields a mean field solution $\langle\hat{\mathcal{V}}(x_p)\rangle_{(3.59)}$ for the composed gauge field variable $\mathcal{V}_{\alpha;\mu}(x_p)$; this classical field solution $\langle\hat{\mathcal{V}}(x_p)\rangle_{(3.59)}$ of (3.59) has to be inserted into the second path integral part of (1.9,3.116) with coset matrix $\hat{T}(x_p)$ (1.10,1.11).

The remaining Fermi determinant comprises the coset matrices $\hat{T}(x_p)$ (1.10,1.11) and the mean field $\langle\hat{\mathcal{V}}(x_p)\rangle_{(3.59)}$ in the thus approximated, one-particle Hamiltonian operator $\langle\hat{\mathcal{H}}(x_p)\rangle_{(3.59)}$ (1.12-1.14); however, this Fermi determinant lacks from a simple, obvious reduction to an action with finite order gradient terms in an effective Lagrangian. Usually, one considers the exponential trace-logarithm form of the determinant where the gradients (and matrix potentials) of the one-particle, mean field operator $\langle\hat{\mathcal{H}}(x_p)\rangle_{(3.59)}$ are weighted by the coset matrices $\hat{T}^{-1}\,\langle\hat{\mathcal{H}}\rangle_{(3.59)}\,\hat{T}$ relative to the eigenvalue spectrum of $\langle\hat{\mathcal{H}}\rangle_{(3.59)}$ which is hence subtracted for a 'relative' gradient operator $\Delta\langle\hat{\mathcal{H}}\rangle_{(3.59)} = \hat{T}^{-1}\,\langle\hat{\mathcal{H}}\rangle_{(3.59)}\,\hat{T} - \langle\hat{\mathcal{H}}\rangle_{(3.59)}$ [3]

$$\Big\langle\hat{\mathcal{H}}^{ba}_{N;M}(y_q,x_p)\Big\rangle_{(3.59)} = \delta^{(4)}(y_q-x_p)\,\eta_q\,\delta_{qp}\begin{pmatrix}\big\langle\hat{H}_{N;M}(x_p)\big\rangle_{(3.59)} & \\ & \big\langle\hat{H}^T_{N;M}(x_p)\big\rangle_{(3.59)}\end{pmatrix}^{ba} \tag{1.12}$$

$$= \delta^{(4)}(y_q-x_p)\,\eta_q\,\delta_{qp}\Big[\hat{\mathcal{B}}\,\hat{\Gamma}^{\mu}\,\hat{S}\,\hat{\partial}_{p,\mu} + \imath\,\hat{\mathcal{B}}\,\hat{\Gamma}^{\mu}\,\hat{\mathfrak{T}}_{\alpha}\,\big\langle\mathcal{V}_{\alpha;\mu}(x_p)\big\rangle_{(3.59)} + \hat{\mathcal{B}}\,\hat{M} - \imath\,\varepsilon_p\,\hat{1}_{2N_0\times 2N_0}\Big];$$

$$\big\langle\hat{H}(x_p)\big\rangle_{(3.59)} = \Big[\hat{\beta}\Big(\hat{\not{\partial}}_p + \imath\,\big\langle\hat{\not{\mathcal{V}}}(x_p)\big\rangle_{(3.59)} - \imath\,\hat{\varepsilon}_p + \hat{m}\Big)\Big] ; \; (\hat{\varepsilon}_p = \hat{\beta}\,\varepsilon_p = \hat{\beta}\,\eta_p\,\varepsilon_+ \,;\; \varepsilon_+ > 0) ; \tag{1.13}$$

$$= \hat{\beta}\hat{\gamma}^{\mu}\,\hat{\partial}_{p,\mu} + \imath\,\hat{\beta}\hat{\gamma}^{\mu}\,\hat{t}_{\alpha}\,\big\langle\mathcal{V}^{\mu}_{\alpha}(x_p)\big\rangle_{(3.59)} + \hat{\beta}\,\hat{m} - \imath\,\varepsilon_p\,\hat{1}_{N_0\times N_0} ;$$

$$\big\langle\hat{H}^T(x_p)\big\rangle_{(3.59)} = \Big[\hat{\beta}\Big(\hat{\not{\partial}}_p + \imath\,\big\langle\hat{\not{\mathcal{V}}}(x_p)\big\rangle_{(3.59)} - \imath\,\hat{\varepsilon}_p + \hat{m}\Big)\Big]^T \tag{1.14}$$

$$= -(\,\hat{\beta}\hat{\gamma}^{\mu}\,)^T\,\hat{\partial}_{p,\mu} + \imath\,(\,\hat{\beta}\hat{\gamma}^{\mu}\,)^T\,\hat{t}^T_{\alpha}\,\big\langle\mathcal{V}^{\mu}_{\alpha}(x_p)\big\rangle_{(3.59)} + \big(\hat{\beta}\,\hat{m}\big)^T - \imath\,\varepsilon_p\,\hat{1}_{N_0\times N_0} ;$$

$$\Delta\langle\hat{\mathcal{H}}\rangle_{(3.59)} = \hat{T}^{-1}\,\langle\hat{\mathcal{H}}\rangle_{(3.59)}\,\hat{T} - \langle\hat{\mathcal{H}}\rangle_{(3.59)} ; \tag{1.15}$$

$$\mathrm{TR}\,\mathfrak{tr}\Big[\ln\big(\hat{T}^{-1}\,\langle\hat{\mathcal{H}}\rangle_{(3.59)}\,\hat{T}\,\big)\Big] = \mathrm{TR}\,\mathfrak{tr}\Big[\ln\Big(\underbrace{\big(\hat{T}^{-1}\,\langle\hat{\mathcal{H}}\rangle_{(3.59)}\,\hat{T} - \langle\hat{\mathcal{H}}\rangle_{(3.59)}\,\big)}_{\Delta\langle\hat{\mathcal{H}}\rangle_{(3.59)}} + \langle\hat{\mathcal{H}}\rangle_{(3.59)}\Big)\Big] ; \tag{1.16}$$

$$\mathrm{TR}\,\mathfrak{tr}\Big[\ln\Big(\Delta\langle\hat{\mathcal{H}}\rangle_{(3.59)} + \langle\hat{\mathcal{H}}\rangle_{(3.59)}\Big)\Big] = \mathrm{TR}\,\mathfrak{tr}\Big[\ln\Big(\hat{1} + \Delta\langle\hat{\mathcal{H}}\rangle_{(3.59)}\,\langle\hat{\mathcal{H}}\rangle^{\mathbf{-1}}_{(3.59)}\Big) + \ln\big(\langle\hat{\mathcal{H}}\rangle_{(3.59)}\big)\Big] . \tag{1.17}$$

Nonetheless, one has to take into account the additional propagation with the inverse of the anomalous doubled one-particle operator $\langle\hat{\mathcal{H}}\rangle^{\mathbf{-1}}_{(3.59)}$ so that the logarithm $\mathrm{TR}\,\mathfrak{tr}[\ln(\hat{1} + \Delta\langle\hat{\mathcal{H}}\rangle_{(3.59)}\,\langle\hat{\mathcal{H}}\rangle^{\mathbf{-1}}_{(3.59)})]$ for the gradient expansion is in fact

[3] In the following the abbreviated trace symbols 'TR', '$\mathfrak{tr}$' denote the summations over the 'Keldysh' 3+1 spacetime contour with inclusion of the anomalous doubled space and over the internal spaces of isospin (flavour), Dirac gamma matrices and colour degrees of freedom. These spaces are specified in detail in appendix A (A.25) and in section 2 (2.12-2.14).

equivalent to the relation TR $\mathfrak{tr}[\ln(\hat{T}^{-1}\,\langle\hat{\mathcal{H}}\rangle_{(3.59)}\,\hat{T}\,\langle\hat{\mathcal{H}}\rangle^{\mathbf{-1}}_{(3.59)})]$

$$\mathrm{TR}\;\mathfrak{tr}\Big[\ln\Big(\hat{1}+\Delta\langle\hat{\mathcal{H}}\rangle_{(3.59)}\;\langle\hat{\mathcal{H}}\rangle^{\mathbf{-1}}_{(3.59)}\Big)\Big] = \mathrm{TR}\;\mathfrak{tr}\Big[\ln\Big(\hat{1}+\big(\hat{T}^{-1}\,\langle\hat{\mathcal{H}}\rangle_{(3.59)}\,\hat{T}-\langle\hat{\mathcal{H}}\rangle_{(3.59)}\big)\,\langle\hat{\mathcal{H}}\rangle^{\mathbf{-1}}_{(3.59)}\Big)\Big] = \mathrm{TR}\;\mathfrak{tr}\Big[\ln\Big(\hat{T}^{-1}\,\langle\hat{\mathcal{H}}\rangle_{(3.59)}\,\hat{T}\,\langle\hat{\mathcal{H}}\rangle^{\mathbf{-1}}_{(3.59)}\Big)\Big]\,. \tag{1.18}$$

The combined occurrence of $\langle\hat{\mathcal{H}}\rangle_{(3.59)}$ and of its inverse $\langle\hat{\mathcal{H}}\rangle^{\mathbf{-1}}_{(3.59)}$, weighted by the coset matrices $\hat{T}^{-1}$, $\hat{T}$, is suggestive of a gradient expansion with large orders for slowly varying coset matrices or BCS terms in the logarithm.

We emphasize this point by a gauge transformation of the coset decomposition so that the mean field operator $\langle\hat{\mathcal{H}}\rangle_{(3.59)}$ with $\langle\hat{\mathcal{V}}(x_p)\rangle_{(3.59)}$ simplifies to a pure gradient operator $\langle\hat{\mathcal{H}}_{\mathfrak{W}}\rangle_{(3.59)}=(\,\langle\hat{H}_{\mathfrak{W}}\rangle_{(3.59)}\,,\,\langle\hat{H}^{T}_{\mathfrak{W}}\rangle_{(3.59)}\,)$ with spatially dependent gamma matrices (section 4.3)

$$\langle\hat{H}_{\mathfrak{W}}\rangle_{(3.59)}=\Big\langle\hat{\beta}_{\mathfrak{W}}(x_p)\;\hat{\gamma}^{\mu}_{\mathfrak{W}}(x_p)\;\hat{\boldsymbol{\partial}}_{\boldsymbol{p},\boldsymbol{\mu}}-\imath\,\varepsilon_p\Big\rangle_{(3.59)}\,. \tag{1.19}$$

Furthermore, the logarithm in (1.18) takes the equivalent gauge transform

$$\mathrm{TR}\;\mathfrak{tr}\Big[\ln\Big(\hat{1}+\Delta\langle\hat{\mathcal{H}}_{\mathfrak{W}}\rangle_{(3.59)}\;\langle\hat{\mathcal{H}}_{\mathfrak{W}}\rangle^{\mathbf{-1}}_{(3.59)}\Big)\Big]=\mathrm{TR}\;\mathfrak{tr}\Big[\ln\Big(\hat{1}+\big(\hat{T}^{-1}\,\langle\hat{\mathcal{H}}_{\mathfrak{W}}\rangle_{(3.59)}\,\hat{T}-\langle\hat{\mathcal{H}}_{\mathfrak{W}}\rangle_{(3.59)}\big)\,\langle\hat{\mathcal{H}}_{\mathfrak{W}}\rangle^{\mathbf{-1}}_{(3.59)}\Big)\Big] = \mathrm{TR}\;\mathfrak{tr}\left[\ln\left(\hat{T}^{-1}\begin{pmatrix}\langle\hat{H}_{\mathfrak{W}}\rangle_{(3.59)} & \\ & \langle\hat{H}_{\mathfrak{W}}\rangle^{T}_{(3.59)}\end{pmatrix}\hat{T}\begin{pmatrix}\langle\hat{H}_{\mathfrak{W}}\rangle^{\mathbf{-1}}_{(3.59)} & \\ & \langle\hat{H}_{\mathfrak{W}}\rangle^{\boldsymbol{T;-1}}_{(3.59)}\end{pmatrix}\right)\right]. \tag{1.20}$$

If one assumes slowly varying finite order gradients of $\hat{T}^{-1}\,\langle\hat{\mathcal{H}}_{\mathfrak{W}}\rangle_{(3.59)}\,\hat{T}$, one will also obtain unintented, extraordinary large spatial and time-like variations with $\hat{T}^{-1}\,\langle\hat{\mathcal{H}}_{\mathfrak{W}}\rangle^{\mathbf{-1}}_{(3.59)}\,\hat{T}$ according to the additional trace operation on the logarithm. In order to circumvent this problem, we suggest the particular integral representation (1.21-1.24) for the logarithm of an operator $\hat{\mathfrak{O}}$ (and similarly for the inverse) in order to approximate the total logarithm with the (coset matrix weighted) combination of $\langle\hat{\mathcal{H}}\rangle_{(3.59)}$ ($\langle\hat{\mathcal{H}}_{\mathfrak{W}}\rangle_{(3.59)}$) and its inverse $\langle\hat{\mathcal{H}}\rangle^{\mathbf{-1}}_{(3.59)}$ ($\langle\hat{\mathcal{H}}_{\mathfrak{W}}\rangle^{\mathbf{-1}}_{(3.59)}$) to simpler actions in an exponential [14]

$$\big(\ln\hat{\mathfrak{O}}\big) = \left(\int_0^{+\infty}dv\quad\frac{\exp\{-v\,\hat{1}\}-\exp\{-v\,\hat{\mathfrak{O}}\}}{v}\right); \tag{1.21}$$

$$\big(\hat{\mathfrak{O}}^{\mathbf{-1}}\big) = \left(\int_0^{+\infty}dv\quad\exp\{-v\,\hat{\mathfrak{O}}\}\right); \tag{1.22}$$

$$\hat{\mathfrak{O}}_{\widetilde{\mathfrak{J}}} = \left(\hat{1}+\Big(\Delta\langle\hat{\mathcal{H}}\rangle_{(3.59)}+\widetilde{\mathfrak{J}}(\hat{T}^{-1},\hat{T})\Big)\,\Big\langle\hat{\mathcal{H}}\Big\rangle^{\mathbf{-1}}_{(3.59)}\right) = \left(\hat{T}^{-1}\;\big\langle\hat{\mathcal{H}}\;\big\rangle_{(3.59)}\hat{T}\;\;\langle\hat{\mathcal{H}}\rangle^{\mathbf{-1}}_{(3.59)}+\widetilde{\mathfrak{J}}(\hat{T}^{-1},\hat{T})\;\langle\hat{\mathcal{H}}\rangle^{\mathbf{-1}}_{(3.59)}\right); \tag{1.23}$$

$$\mathcal{A}_{DET}[\hat{T},\langle\hat{\mathcal{V}}\rangle_{(3.59)};\hat{\mathcal{J}}\equiv 0] = \frac{1}{2}\int_0^{+\infty}dv\quad\underset{\int_C d^4x_p\;\eta_p}{\overset{a(=1,2)}{\mathrm{TR}}}\;\;\underset{N_f,\hat{\gamma}^{(\mu)}_{mn},N_c}{\mathfrak{tr}}\left[\frac{\exp\big\{-v\,\hat{1}\big\}-\exp\Big\{-v\,\hat{T}^{-1}\big\langle\hat{\mathcal{H}}\big\rangle_{(3.59)}\hat{T}\;\langle\hat{\mathcal{H}}\rangle^{\mathbf{-1}}_{(3.59)}\Big\}}{v}\right]. \tag{1.24}$$

If we suppose positive eigenvalues at order unity or far beyond for the total operator $\hat{\mathfrak{O}}=\hat{T}^{-1}\,\langle\hat{\mathcal{H}}\rangle_{(3.59)}\,\hat{T}\,\langle\hat{\mathcal{H}}\rangle^{\mathbf{-1}}_{(3.59)}$, the inverse factorials $1/n!$ of $\exp\{-v\,\hat{T}^{-1}\,\langle\hat{\mathcal{H}}\rangle_{(3.59)}\,\hat{T}\,\langle\hat{\mathcal{H}}\rangle^{\mathbf{-1}}_{(3.59)}\,\}$ cause a rapid, meaningful expansion and convergence instead of a pure logarithm $\ln(\,\hat{T}^{-1}\,\langle\hat{\mathcal{H}}\rangle_{(3.59)}\,\hat{T}\,\langle\hat{\mathcal{H}}\rangle^{\mathbf{-1}}_{(3.59)}\,)$ with reciprocal integer numbers in the expansion. Therefore, we rely on the integral representation (1.21-1.24) of the logarithm and of the inverse of an operator $\hat{\mathfrak{O}}$ and apply these relations for reducing the path integral part to effective actions with coset matrices $\hat{T}(x_p)$ and anti-symmetric coset generator $\hat{X}(x_p)$, $\hat{X}^{\dagger}(x_p)$ for BCS quark pairs. We can even choose the eigenbasis of the mean field approximated, one-particle operator $\langle\hat{H}\rangle_{(3.59)}$ or of its anomalous doubled version $\langle\hat{\mathcal{H}}\rangle_{(3.59)}$ instead of the 3+1 dimensional coordinate representation. This

particular matrix representation for $\hat{T}$ in terms of the eigenbasis of $\langle\hat{\mathcal{H}}\rangle_{(3.59)}$ allows to calculate observables as correlation functions of anomalous quark field combinations $\langle\psi_M(x_p)\;\psi_N(x_p)\rangle$, density terms $\langle\psi_M^*(x_p)\;\psi_N(x_p)\rangle$ and normalized eigenvalue correlations of BCS terms originating from non-Abelian gauge theories as QCD.

Using the above mentioned gauge invariance of coset matrices and generators in the coset decomposition, a kind of *'interaction'* representation allows to transform the mean field operator $\langle\hat{\mathcal{H}}\rangle_{(3.59)}$ to the already mentioned, pure gradient terms $\langle\hat{\mathcal{H}}_{\hat{\mathfrak{W}}}\rangle_{(3.59)}$ with spatially varying gamma matrices (1.19). One can take this particular 'interaction' representation in order to extract a *Hopf invariant* with one-form $\hat{\omega}_1(x_p)$

$$N_I=\frac{1}{V_{S^{2n-1}}}\int_{S^{2n-1}}\hat{\omega}_{n-1}\wedge(d\hat{\omega}_{n-1})\;\;;\;\;(\hat{\omega}_{n-1}=\text{n-1 form with n=2,4,8!})\;;\;(V_{S^{2n-1}}=\text{volume of }S^{2n-1}\text{ sphere})\;,\quad(1.25)$$

from the axial current conservation as a nontrivial topology for the Hopf mapping $\Pi_3(S^2)=\mathsf{Z}$ [15, 16]. The zero component of the axial current (with chiral anomaly) contains such a Hopf invariant where one has to consider the mapping from the 3D spatial coordinate space to the internal S^2 sphere with quaternion-valued, anti-symmetric Pauli matrix $(\hat{\tau}_2)_{gf}$ as eigenvalues with corresponding complex, even-valued isospin field $\bar{f}_r(x_p)$ for BCS quark pairs (Compare for a derivation of the chiral anomaly from the ordinary QCD path integral with appendix C and with Refs. [33, 34]). Thus our strictly derived BCS path integral is more closely related to a Skyrme-Faddeev string model [8] with Hopf mapping $\Pi_3(S^2)=\mathsf{Z}$ than to a topological Skyrme Lagrangian with baryons as winding numbers following from $\Pi_3(\mathrm{SU}(2))=\Pi_3(S^3)=\mathsf{Z}$ [17, 18, 19]. Although finite order gradient expansions have to be taken at least up to fourth order for stable energy configurations and are questionable concerning the validity of low-momentum approximations, we describe in appendix D various principles which have necessarily to be regarded for an appropriate expansion in the anomalous doubled Hilbert space of quantum many particle physics. The remaining coset matrices for the BCS degrees of freedom propagate with anomalous doubled Green functions, containing density related background fields, and are applied to compose an effective (Skyrme-like-)Lagrangian; this 'analogous' Skyrme-like Lagrangian follows from the gradient expansion of the self-energy operator within the fermi-determinant and within its inverse of bilinear source fields, but qualitatively differs by the derived Hopf invariant $\Pi_3(S^2)=\mathsf{Z}$ from the original Skyrme Lagrangian with homotopy mapping $\Pi_3(\mathrm{SU}(2))=\Pi_3(S^3)=\mathsf{Z}$.

1.2 Symmetry breaking source fields for mesons and baryons

In order to generate observables from the various, different kinds of path integrals, we introduce the even-valued, in general nonlocal source matrix $\hat{\mathcal{J}}_{N;M}^{ba}(y_q,x_p)$ with two spacetime arguments y_q, x_p on the Keldysh time contour and with anomalous indexing $a,b=1,2$ and internal space indices M,N to be specified in relations (2.7-2.14). This general source field tracks the original observables in terms of quark fields in the ordinary path integral (2.23-2.27) through multiple transformations (as HST's) to path integrals with self-energies as the remaining field degrees of freedom. As one considers an observable $\langle\Psi_N^{\dagger,b}(y_q)\;\Psi_M^a(x_p)\rangle$ of bilinear quark fields by differentiating the original, ordinary QCD path integral with respect to this source $\hat{\mathcal{J}}_{N;M}^{ba}(y_q,x_p)$, one can also generate this same observable for quark fields later after several HST's and a coset decomposition just by taking the same differentiation of the final transformed generating function.

Besides we incorporate a symmetry breaking, odd-valued source field $j_{\psi;M}(x_p)$ (or its anomalous doubled version $J_{\psi;M}^{a(=1,2)}(x_p)$) which causes non-vanishing observables of quark fields in odd number as $\langle\psi_M(x_p)\;\psi_N(x_p)\;\psi_{N'}(x_p)\rangle$ or $\langle\psi_M(x_p)\rangle$, etc.. These source fields $j_{\psi;M}(x_p)$ are important for the fermionic degrees in the nuclei and may lead to an analogous coherent, but fermionic wavefunction as in a BE-condensation for a macroscopic, coherent wavefunction. Furthermore, one has to include a symmetry breaking source matrix $\hat{j}_{\psi\psi;N;M}(x_p)$, $\hat{j}_{\psi\psi;N;M}^{\dagger}(x_p)$ (or its combined anomalous doubled version $\hat{J}_{\psi\psi;N;M}^{ba}(x_p)$) for creating an initial configuration of BCS quark pairs for the strong spin pairing force. However, we do not regard a detailed phase transition from incoherent initial conditions to a coherent configuartion of BCS quark pairs; this would involve a phase transition with a detailed, experimental dependence on temperature, density, etc.. Therefore, we just set an initial, coherent configuration of BCS terms in the coset matrix $\hat{T}(x_p)$ and generators $\hat{X}(x_p)$, $\hat{X}^{\dagger}(x_p)$ at intermediate times with wave-packets of appropriate space- and momentum-dependence, neglecting a detailed phase transition from 'incoherence' to 'coherence' at earlier times $t_p\rightarrow-\infty$.

2 The path integral with symmetry breaking source fields

2.1 Definitions

The derivation for the effective BCS terms begins with the standard QCD-type Lagrangian (2.1-2.6) of anti-commuting quark field spinors $\psi(x)$, $\psi^{\dagger}(x)$ and with the non-Abelian $\mathrm{SU}_c(N_c=3)$ gauge fields $A_{\alpha;\mu}(x)$ and the corresponding field strength tensor $\hat{F}_{\alpha;\mu\nu}(x)$. We take the notations defined in [20, 21] ("The Quantum Theory of Fields" Vol. 1-2, S. Weinberg) and label the eight $\mathrm{SU}_c(N_c=3)$ gauge field degrees of freedom $A_{\alpha;\mu}(x)$ by the first Greek letters $\alpha,\,\beta,\,\gamma,\ldots=1,\ldots,8$ which are separated from the 3+1 spacetime or Lorentz-indices of the middle of the Greek alphabet ($\kappa,\,\lambda,\,\mu,\,\nu=0,1,2,3$) by a semicolon (spacetime metric tensor $\hat{\eta}^{\mu\nu}$ with $\hat{\eta}^{00}=-1$ and $\hat{\eta}^{ij}=+\delta^{ij}$, $i,j,k,\ldots=1,2,3$). The covariant derivative $\hat{D}_\mu$ (2.3) is defined in anti-hermitian kind with spacetime derivative $\hat{\partial}_\mu$ and additional imaginary factor '$\imath$' which is attached to the eight hermitian $\mathrm{SU}_c(N_c=3)$ 'colour' generators $\hat{t}_\alpha$ in the fundamental representation with totally anti-symmetric structure constants $C_{\alpha\beta\gamma}$. We assume that all physical quantities are scaled to corresponding dimensionless objects and list the entire Lagrangian $\mathcal{L}(\psi,\,\hat{D}_\mu\psi,\,\hat{F})$ (2.1) with gauge field strength tensor $\hat{F}_{\alpha;\mu\nu}(x)$ (2.4,2.5) and coupled fermionic-matter Lagrangian $\mathcal{L}_M(\psi,\hat{D}_\mu\psi)$ (2.2) according to Ref. [20, 21] ("The Quantum Theory of Fields" Vol. 2, S. Weinberg), ($\hat{m}_f=\mathrm{diag}\{m_u,\,m_d,\,(m_s)\}$)

$$
\begin{aligned}
\mathcal{L}(\psi,\,\hat{D}_\mu\psi,\,\hat{F}) &= -\frac{1}{4}\,\hat{F}_{\alpha;\mu\nu}(x)\,\hat{F}_\alpha^{\mu\nu}(x)+\mathcal{L}_M(\psi,\hat{D}_\mu\psi)\;; &(2.1)\\
\mathcal{L}_M(\psi,\hat{D}_\mu\psi) &= -\sum_{f=u,d,(s)}\bar{\psi}_f(x)\,\Big[\hat{\not{D}}+\hat{m}_f\Big]\,\psi_f(x)=-\sum_{f=u,d,(s)}\bar{\psi}_f(x)\,\Big[\hat{\not{\partial}}-\imath\hat{\not{A}}(x)+\hat{m}_f\Big]\,\psi_f(x) &(2.2)\\
&= -\sum_{f=u,d,(s)}\bar{\psi}_f(x)\,\Big[\hat{\gamma}^\mu\,\hat{\partial}_\mu-\imath\,\hat{\gamma}^\mu\,A_{\alpha;\mu}(x)\,\hat{t}_\alpha+\hat{m}_f\Big]\,\psi_f(x)\;;\\
\hat{D}_\mu\psi(x) &= \hat{\partial}_\mu\psi(x)-\imath\,\hat{t}_\alpha\,A_{\alpha;\mu}(x)\,\psi(x)\;;\quad \hat{\not{D}}(x)=\hat{\gamma}^\mu\,\hat{D}_\mu(x)\;;\quad \hat{\not{\partial}}=\hat{\gamma}^\mu\,\hat{\partial}_\mu\;; &(2.3)\\
\Big[\hat{D}_\mu(x)\,,\,\hat{D}_\nu(x)\Big] &= -\imath\,\hat{t}_\alpha\,\hat{F}_{\alpha;\mu\nu}(x)\;; &(2.4)\\
\hat{F}_{\alpha;\mu\nu}(x) &= \hat{\partial}_\mu A_{\alpha;\nu}(x)-\hat{\partial}_\nu A_{\alpha;\mu}(x)+C_{\alpha\beta\gamma}\,A_{\beta;\mu}(x)\,A_{\gamma;\nu}(x)\;; &(2.5)\\
\hat{\not{A}}(x) &= \hat{\not{A}}_\alpha(x)\,\hat{t}_\alpha=\hat{\gamma}^\mu\,A_{\alpha;\mu}(x)\,\hat{t}_\alpha\;. &(2.6)
\end{aligned}
$$

In the following we distinguish between three internal spaces whose independent degrees of freedom are marked by the indices listed in Eqs. (2.7-2.9). The first part of these three independent internal spaces is determined to be the $\mathrm{SU}_f(N_f=2)$ isospin (or the extended $\mathrm{SU}_f(N_f=3)$ flavour) space which is labelled by the Latin letters $f,\,g,\ldots$ with further sub-indexing $f_1,\,g_1,\ldots$ and primes $f',\,g',\ldots$ for 'up', 'down', ('strange') quarks. The general quark matter Lagrangian is assumed with non-degenerate isospin (or flavour) masses $m_f,\,m_{f'},\ldots,\,m_g,\,m_{g'},\ldots$ which are diagonal in the Dirac-gamma matrices and $\mathrm{SU}_c(N_c=3)$ colour space. We also apply the Clifford-algebra for four-dimensional spacetime with 4×4 Dirac matrices $\hat{\gamma}^\mu_{mn}$ of Ref. [20] and define the indices $m,\,n,\,m',\,n',\,m_1,\,n_1,\ldots$ with further sub-indexing and primes for particular matrix elements of these 4×4 gamma matrices as $\hat{\gamma}^\mu_{mn}$ and $(\hat{\gamma}_5)_{mn}=-\imath\,(\hat{\gamma}^0\hat{\gamma}^1\hat{\gamma}^2\hat{\gamma}^3)_{mn}$. Apart from the Greek indices $\alpha,\,\beta,\,\gamma$ for labelling the eight $\mathrm{SU}_c(N_c=3)$ generators $\hat{t}_\alpha$, the indices $r,\,s$ (with further sub-indexing) are ascribed to the 3×3 fundamental matrix representation $\hat{t}_{\alpha;rs}$ of $\mathrm{SU}_c(N_c=3)$ colour degrees of freedom

$$
\begin{aligned}
&\text{isospin (or flavour) index} &:&\quad f,\,f',\,f_1,\ldots,\,g,\,g',\,g_1,\ldots=\mathrm{u(p),\,d(own),\,(s(trange))}; &(2.7)\\
&\mathrm{SU}_f(2)\text{ isospin }(\mathrm{SU}_f(3)\text{ flavour) mass-matrices} &:&\quad m_f,\,m_{f'},\,m_{f_1},\ldots,\,m_g,\,m_{g'},\,m_{g_1},\ldots\;;\\
&\text{indices for gamma matrices }\hat{\gamma}^\mu_{4\times4}=\hat{\gamma}^\mu_{mn} &:&\quad m,\,m',\,m_1,\ldots,\,n,\,n',\,n_1,\ldots=1,\ldots,4\;; &(2.8)\\
&\text{indices of }\mathrm{SU}_c(N_c=3)\text{ colour matrices }\hat{t}_{\alpha;rs} &:&\quad \alpha,\,\beta,\,\gamma,\,\alpha',\,\beta',\,\gamma',\,\alpha_1,\,\beta_1,\,\gamma_1,\ldots=1,\ldots,8\;; &(2.9)\\
& &:&\quad r,\,s,r',\,s',\,r_1,\,s_1,\ldots=1,2,3\;.
\end{aligned}
$$

In consequence we attribute the three internal spaces to the Grassmann-valued quark spinors (2.10) by using the labels $\{f,m,r\}$, $\{g,n,s\}$, ... for the isospin- (flavour-) matrices , gamma-matrices and colour-matrix degrees of freedom

$$
\psi(x):=\psi_{f,m,r}(x)\quad;\quad \bar{\psi}_{g,n,s}(x):=\psi^\dagger_{g,n',s}(x)\,\hat{\beta}_{n'n}\;;\quad \hat{\beta}=\imath\,\hat{\gamma}^0\;. \tag{2.10}
$$

In order to simplify notations, we combine the isospin- (flavour-) indices f, g, ..., gamma-matrix indices m, n, ... and colour-matrix indices r, s, ... to the collective uppercase indices $M=\{f,m,r\}$, $N=\{g,n,s\}$, ... with further possible sub-indexing. Since we particularly specify on chiral symmetry transformations, the isospin (flavour) index f or g is partially separated from the remaining gamma- and colour-matrix indices which are combined and abbreviated by a bar over the corresponding uppercase letters as $\overline{M}$, $\overline{N}$, ..., $\overline{M}_1$, $\overline{N}_1$, ... Therefore, the entire, collective indices M, N, M', N', ... can also be indicated by the combination of the isospin (flavour) index f, g, ... and remaining collective index $\overline{M}$, $\overline{N}$, ... for gamma- and colour-matrices

$$\begin{array}{rclrclrclc} M &:=& \{f,m,r\} & M' &:=& \{f',m',r'\} & M_1 &:=& \{f_1,m_1,r_1\} & \ldots \\ N &:=& \{g,n,s\} & N' &:=& \{g',n',s'\} & N_1 &:=& \{g_1,n_1,s_1\} & \ldots \\ \overline{M} &:=& \{m,r\} & \overline{M}' &:=& \{m',r'\} & \overline{M}_1 &:=& \{m_1,r_1\} & \ldots \\ \overline{N} &:=& \{n,s\} & \overline{N}' &:=& \{n',s'\} & \overline{N}_1 &:=& \{n_1,s_1\} & \ldots \\ M &:=& \{f,\overline{M}\} & M' &:=& \{f',\overline{M}'\} & M_1 &:=& \{f_1,\overline{M}_1\} & \ldots \\ N &:=& \{g,\overline{N}\} & N' &:=& \{g',\overline{N}'\} & N_1 &:=& \{g_1,\overline{N}_1\} & \ldots \end{array} . \tag{2.11}$$

The final effective BCS related Lagrangian is extracted by performing traces over various combinations of these three internal spaces. For that reason we have to distinguish between the various traces listed in Eq. (2.12). We denote the symbols $\mathfrak{tr}_{N_f}$, $\mathfrak{tr}_{\hat{\gamma}_{mn}^{(\mu)}}$, $\mathfrak{tr}_{N_c}$ for taking trace operations over isospin (flavour) matrices, gamma-matrices and $\mathrm{SU}_c(N_c=3)$ colour matrices according to the above list of indices and labels in Eqs. (2.7-2.11)

$$\underset{N_f}{\mathfrak{tr}}\Big[\ldots\Big]\;;\quad \underset{\hat{\gamma}_{mn}^{(\mu)}}{\mathfrak{tr}}\Big[\ldots\Big]\;;\quad \underset{N_c}{\mathfrak{tr}}\Big[\ldots\Big]\,. \tag{2.12}$$

In the remainder combinations of the traces (2.12) for these separate internal spaces follow straightforwardly and will be abbreviated as in Eq. (2.13), as we proceed to the final form of the effective actions for BCS quark pair condensates

$$\underset{N_f,\hat{\gamma}_{mn}^{(\mu)}}{\mathfrak{tr}}\Big[\ldots\Big]\;;\quad \underset{N_f,N_c}{\mathfrak{tr}}\Big[\ldots\Big]\;;\quad \underset{\hat{\gamma}_{mn}^{(\mu)},N_c}{\mathfrak{tr}}\Big[\ldots\Big]\;;\quad \underset{N_f,\hat{\gamma}_{mn}^{(\mu)},N_c}{\mathfrak{tr}}\Big[\ldots\Big]\,. \tag{2.13}$$

Apart from the above traces (2.12,2.13), we point out the overall trace $\mathfrak{Tr}^a_{N_f,\hat{\gamma}_{mn}^{(\mu)},N_c}$ (2.14) of *all internal spaces*, which includes an additional summation with a, b, c, ... $=1,2$ over the anomalous doubled space for BCS pair condensates. In comparison to this overall trace (2.14), the above listed traces (2.12,2.13) only encompass density terms without any possible summations over anomalous pairings of quark fields

$$\underset{N_f,\hat{\gamma}_{mn}^{(\mu)},N_c}{\overset{a(=1,2)}{\mathfrak{Tr}}}\Big[\ldots\Big]\quad . \tag{2.14}$$

2.2 The path integral with fermionic matter- and non-Abelian gauge fields

According to the entire Lagrangian $\mathfrak{L}(\psi,\,\hat{D}_\mu\psi,\,\hat{F})$ (2.1), we construct the analogous path integral with fermionic quark matter and non-Abelian gauge fields. However, we introduce the time contour integral (2.15) with time variable $x^0_{p=\pm}$ on the two branches $p=\pm$ for time development with $\mathfrak{L}$, $\mathfrak{L}_M$ (2.1,2.2) in forward $\int_{-\infty}^{+\infty}dx_+^0\ldots$ and backward $\int_{+\infty}^{-\infty}dx_-^0\ldots$ direction [11, 12, 22, 23]. The negative sign of the backward propagation $\int_{+\infty}^{-\infty}dx_-^0\ldots=-\int_{-\infty}^{+\infty}dx_-^0\ldots$ will be frequently taken into account by the symbol $\eta_{p=\pm}=p=\pm$ (2.16) as a contour time metric [4]

$$\int_C d^4x_p\;\ldots\;=\;\int_{L^3}d^3\vec{x}\bigg(\int_{-\infty}^{+\infty}dx_+^0\;\ldots+\int_{+\infty}^{-\infty}dx_-^0\;\ldots\bigg)=\int_{L^3}d^3\vec{x}\bigg(\int_{-\infty}^{+\infty}dx_+^0\;\ldots-\int_{-\infty}^{+\infty}dx_-^0\;\ldots\bigg)$$

[4] This contour time metric $\eta_p=p=\pm$ should not be confused with the spacetime metric tensor $\eta^{\mu\nu}$ ($\eta^{00}=-1$, $\eta^{ij}=\delta^{ij}$) for contravariant and covariant components of vectors and tensors. The indices p, q, p', q', ... are reserved for the contour time metric η_p, η_q, $\eta_{p'}$, $\eta_{q'}$, ... of the two branches for forward and backward propagation; on the contrary the indices κ, λ, μ, ν from the middle of the Greek alphabet are Lorentz-indices of four-dimensional spacetime, as e. g. for the metric tensor $\eta^{\mu\nu}$, $\eta^{\kappa\lambda}$, x^μ, x_ν, ...

$$
\begin{aligned}
&= \int_{L^3} d^3\vec{x}\bigg(\sum_{p=\pm}\int_{-\infty}^{+\infty} dx_p^0\;\eta_p\;\ldots\bigg)\;; && (2.15)\\
\eta_p &= \{\underbrace{+1}_{p=+}\;;\;\underbrace{-1}_{p=-}\}\quad . && (2.16)
\end{aligned}
$$

An additional contour time label p, q, ... of the four-dimensional vector $x^{\mu}=(x^0,\vec{x})\rightarrow x_p^{\mu}=(x_p^0,\vec{x})$ refers to the two different propagations of zero components $x_{\pm}^0$ of these contour-time extended four-vectors x_p

$$
\begin{aligned}
x_p^{\mu} &= \big(x_p^0\,,\,\vec{x}\big)\;; & x_+^{\mu} &= \big(x_+^0\,,\,\vec{x}\big)\;; & x_-^{\mu} &= \big(x_-^0\,,\,\vec{x}\big)\;; \\
\hat{\partial}_{p,\mu} &= \bigg(\frac{\partial}{\partial x_p^0}\,,\,\frac{\partial}{\partial\vec{x}}\bigg)\;; & \hat{\partial}_{+,\mu} &= \bigg(\frac{\partial}{\partial x_+^0}\,,\,\frac{\partial}{\partial\vec{x}}\bigg)\;; & \hat{\partial}_{-,\mu} &= \bigg(\frac{\partial}{\partial x_-^0}\,,\,\frac{\partial}{\partial\vec{x}}\bigg)\,.
\end{aligned}
\qquad (2.17)
$$

We consider four different source fields on the non-equilibrium time contour where two of these are applied for a spontaneous symmetry breaking of the fermionic matter fields. The general, anomalous doubled, complex, even-valued source term $\hat{\jmath}_{g,n,s;f,m,r}^{ba}(y_q,x_p)$ allows to generate observables of bilinear, quartic or higher order, even-numbered quark fields including BCS terms as $\langle\psi_{g,n,s}(y_q)\;\psi_{f,m,r}(x_p)\rangle$. However, we discern between the source $\hat{\jmath}_{N;M}^{ba}(y_q,x_p)$ for generating even-numbered quark field observables by differentiating a path integral and the symmetry breaking, even-valued, complex field $\hat{J}_{\psi\psi;N;M}^{b\neq a}(x_p)$ which couples to $\psi_{g,n,s}^{*}(x_p)\;\psi_{f,m,r}^{*}(x_p)$ and $\psi_{g,n,s}(x_p)\;\psi_{f,m,r}(x_p)$. In order to create non-vanishing quark fields in odd number, a fermionic, Grassmann-valued source $J_{\psi;f,m,r}^{a}(x_p)$ is incorporated which couples to single quark fields $\psi_{f,m,r}(x_p)$, $\psi_{f,m,r}^{\dagger}(x_p)$ and that is also extended with its complex value to an anomalous doubled form. Furthermore, we allow for an anti-symmetric, even-valued, *real* field $\hat{\mathrm{j}}_{\alpha}^{(\hat{F})\mu\nu}(x_p)$ which generates the non-Abelian gauge field strength tensor $\hat{F}_{\alpha}^{\mu\nu}(x_p)$. Since the gauge fields are changed to background fields in later steps of the derivation to the final effective Lagrangian, we do not take into account a symmetry breaking of the gauge field strength tensor or an anomalous doubling as for the quark fields and simply set the source $\hat{\mathrm{j}}_{\alpha}^{(\hat{F})\mu\nu}(x_p)$ for the field strength to zero at the end of the calculation

$$
\begin{array}{rclrcl}
\multicolumn{3}{c}{\text{Source fields}\quad :} & & & \\
\hat{\jmath}_{g,n,s;f,m,r}^{ba}(y_q,x_p) & \in & \mathsf{C}_{\mathsf{even}}\;; & J_{\psi;f,m,r}^{a}(x_p) & \in & \mathcal{C}_{odd}\;; \\
\hat{J}_{\psi\psi;g,n,s;f,m,r}^{b\neq a}(x_p) & \in & \mathsf{C}_{\mathsf{even}}\;; & \hat{\mathrm{j}}_{\alpha;\mu\nu}^{(\hat{F})}(x_p) & \in & \mathsf{R}_{\mathsf{even}}\quad .
\end{array}
\qquad (2.18)
$$

The precise form of the entire symmetry breaking action $\mathcal{A}_S[\hat{\jmath},J_{\psi},\hat{J}_{\psi\psi},\hat{\mathrm{j}}^{(\hat{F})}]$ is given in Eq. (2.22) for all these four source fields in (2.18). In advance we mention that this source action $\mathcal{A}_S[\hat{\jmath},J_{\psi},\hat{J}_{\psi\psi},\hat{\mathrm{j}}^{(\hat{F})}]$ is specified in its anomalous doubled kind with metric $\hat{S}$ (2.21) for the anomalous doubled Fermi fields. A two-component, anomalous doubled, fermionic field $(\ldots)^{a(=1,2)}$ follows from the extension with its complex-valued copy

$$
\begin{aligned}
\big(\text{field}\big) &\rightarrow \text{anomalous doubled} \rightarrow \Big(\ldots\Big)^{a(=1,2)} &:=& \begin{pmatrix}\big(\text{field}\big)^{a=1}\\ \big(\text{field}^{*}\big)^{a=2}\end{pmatrix}\;; \\
\big(\text{field}^{\dagger}\big) &\rightarrow \text{anomalous doubled} \rightarrow \Big(\ldots\Big)^{\dagger,a(=1,2)} &:=& \Big(\big(\text{field}^{*}\big)^{a=1}\;;\;\big(\text{field}\big)^{a=2}\Big)\,.
\end{aligned}
\qquad (2.19)
$$

Since fermionic quark matter fields are only considered for the anomalous doubling, one has to introduce a diagonal negative sign $-\hat{1}_{N;M}$ in the '22' block of the metric tensor $\hat{S}$ for anomalous doubling. We have to apply for this metric tensor $\hat{S}$ the Weyl unitary trick where the metric tensor $\hat{S}$ is factorized into $\hat{I}\cdot\hat{I}=\hat{S}$ with the new metric $\hat{I}=\delta_{ba}\,\{+\hat{1}_{N;M}\;;\;\imath\,\hat{1}_{N;M}\}$. The inverse $\hat{I}^{-1}$ of $\hat{I}$ is given by the product $\hat{I}^{-1}=\hat{S}\,\hat{I}$ which is frequently used in transformations involving the Weyl unitary trick for the coset decomposition

$$
\begin{aligned}
\psi_M^{\dagger}\psi_M &= \frac{1}{2}\Big(\psi_N^{\dagger}\;\hat{1}_{N;M}\;\psi_M-\psi_M^{T}\;\hat{1}_{M;N}\;\psi_N^{*}\Big) && (2.20)\\
&= \frac{1}{2}\Big(\psi_N^{*}\,,\,\psi_N\Big)^{b}\underbrace{\begin{pmatrix}\hat{1}_{N;M} & \\ & -\hat{1}_{N;M}\end{pmatrix}^{ba}}_{\hat{S}^{ba}}\begin{pmatrix}\psi_M\\ \psi_M^{*}\end{pmatrix}^{a}\;;
\end{aligned}
$$

$$\hat{S}^{ba} = \delta_{ba}\Big\{\underbrace{+\hat{1}_{N;M}}_{a=1}\,;\,\underbrace{-\hat{1}_{N;M}}_{a=2}\Big\}\;;\qquad \hat{I}^{ba}=\delta_{ba}\Big\{\underbrace{+\hat{1}_{N;M}}_{a=1}\,;\,\underbrace{\imath\,\hat{1}_{N;M}}_{a=2}\Big\}\;;\qquad \hat{I}\cdot\hat{I}=\hat{S}\;; \tag{2.21}$$

$$\begin{aligned}\mathcal{A}_S[\hat{\jmath},J_\psi,\hat{J}_{\psi\psi},\hat{\mathrm{j}}^{(\hat{F})}] &= \frac{1}{2}\int_C d^4x_p\Big(J^{\dagger,a}_{\psi;m,f,r}(x_p)\;\hat{S}^{ab}\;\Psi^b_{m,f,r}(x_p)+\Psi^{\dagger,a}_{m,f,r}(x_p)\;\hat{S}^{ab}\;J^b_{\psi;m,f,r}(x_p)\Big)+ \\ &+\;\frac{1}{2}\int_C d^4x_p\Big(\psi_{g,n,s}(x_p)\;\hat{j}^\dagger_{\psi\psi;g,n,s;f,m,r}(x_p)\;\psi_{f,m,r}(x_p)+\psi^*_{g,n,s}(x_p)\;\hat{j}_{\psi\psi;g,n,s;f,m,r}(x_p)\;\psi^*_{f,m,r}(x_p)\Big)+ \\ &+\;\int_C d^4x_p\;\hat{\mathrm{j}}^{(\hat{F})}_{\alpha;\mu\nu}(x_p)\;\hat{F}^{\mu\nu}_\alpha(x_p)+\frac{1}{2}\int_C d^4x_p\;d^4y_q\;\Psi^{\dagger,b}_{g,n,s}(y_q)\;\hat{\jmath}^{ba}_{g,n,s;f,m,r}(y_q,x_p)\;\Psi^a_{f,m,r}(x_p)\;.\end{aligned} \tag{2.22}$$

As already mentioned, the source $\hat{\jmath}^{ba}_{g,n,s;f,m,r}(y_q,x_p)$ differs from the other two source fields $\hat{J}^{b\neq a}_{\psi\psi;N;M}(x_p)$, $J^a_{\psi;M}(x_p)$ for the quark fields. The latter are assigned to symmetry breaking processes and therefore are set to equivalent, *non-vanishing* values on the two branches of the time contour at the final end of calculations (2.24); in consequence an entire hermitian action results into a normalized path integral or generating function. [5] However, this normalization to unity requires a vanishing source term $\hat{\jmath}^{ba}_{N;M}(y_q,x_p)$ which tests the response to bilinear quark terms (also of the anomalous case) propagating with Lagarangian $\mathcal{L}(\psi,\hat{D}_\mu\psi,\hat{F})$ (2.1,2.2) and hermitian, symmetry breaking sources $\hat{J}^{b\neq a}_{\psi\psi;N;M}(x_+)=\hat{J}^{b\neq a}_{\psi\psi;N;M}(x_-)$, $J^a_{\psi;M}(x_+)=J^a_{\psi;M}(x_-)$. This response to $\hat{\jmath}^{ba}_{N;M}(y_q,x_p)\neq 0$ can be extended to higher order functional Taylor-expansion of the path integral $Z[\hat{\jmath},J_\psi,\hat{J}_{\psi\psi},\hat{\mathrm{j}}^{(\hat{F})}]$ so that one obtains the response for higher order correlation functions of quark fields in even number (also with inclusion of BCS related terms)

$$\begin{aligned}Z[\hat{\jmath},J_\psi,\hat{J}_{\psi\psi},\hat{\mathrm{j}}^{(\hat{F})}] &= \overbrace{Z[\hat{\jmath}\equiv 0\,,\,J_\psi(x_+)=J_\psi(x_-)\,,\,\hat{J}_{\psi\psi}(x_+)=J_{\psi\psi}(x_-)\,,\,\hat{\mathrm{j}}^{(\hat{F})}\equiv 0]}^{\equiv 1}+ \\ &+\int_C d^4x_p\;d^4y_q\;\hat{\jmath}^{ba}_{N;M}(y_q,x_p)\left(\frac{\delta}{\delta\hat{\jmath}^{ba}_{N;M}(y_q,x_p)}Z[\hat{\jmath},J_\psi,\hat{J}_{\psi\psi},\hat{\mathrm{j}}^{(\hat{F})}]\right)\Bigg|_{\text{conditions}}+ \\ &+\frac{1}{2!}\int_C d^4x^{(1)}_{p_1}\;d^4y^{(1)}_{q_1}\;d^4x^{(2)}_{p_2}\;d^4y^{(2)}_{q_2}\;\;\hat{\jmath}^{b_1a_1}_{N_1;M_1}(y^{(1)}_{q_1},x^{(1)}_{p_1})\;\;\hat{\jmath}^{b_2a_2}_{N_2;M_2}(y^{(2)}_{q_2},x^{(2)}_{p_2})\;\times \\ &\times\left(\frac{\delta}{\delta\hat{\jmath}^{b_1a_1}_{N_1;M_1}(y^{(1)}_{q_1},x^{(1)}_{p_1})}\frac{\delta}{\delta\hat{\jmath}^{b_2a_2}_{N_2;M_2}(y^{(2)}_{q_2},x^{(2)}_{p_2})}Z[\hat{\jmath},J_\psi,\hat{J}_{\psi\psi},\hat{\mathrm{j}}^{(\hat{F})}]\right)\Bigg|_{\text{conditions}}+\frac{1}{3!}\ldots\;;\end{aligned} \tag{2.23}$$

$$\text{conditions} = \Big\{\hat{\jmath}\equiv 0\,,\,J_\psi(x_+)=J_\psi(x_-)\,,\,\hat{J}_{\psi\psi}(x_+)=\hat{J}_{\psi\psi}(x_-)\,,\,\hat{\mathrm{j}}^{(\hat{F})}\equiv 0\Big\}\,. \tag{2.24}$$

According to Ref. [20, 21], we perform the path integral quantization for the Lagrangian $\mathcal{L}(\psi,\hat{D}_\mu\psi,\hat{F})$ (2.1,2.2) with source action $\mathcal{A}_S[\hat{\jmath},J_\psi,\hat{J}_{\psi\psi},\hat{\mathrm{j}}^{(\hat{F})}]$ (2.22) on the time contour in axial gauge which does not contain the Gribov ambiguity. This is accomplished by introducing a space-like, four-component, unit vector $n^\mu=(0,\vec{n})$ $(n_\mu n^\mu=1)$ into a delta-function with the gauge field $A_{\alpha;\mu}(x_p)$

$$\begin{aligned}Z[\hat{\jmath},J_\psi,\hat{J}_{\psi\psi},\hat{\mathrm{j}}^{(\hat{F})}] &= \int d[\psi^\dagger_M(x_p)\,,\,\psi_M(x_p)]\;\;d[A_{\alpha;\mu}(x_p)]\left\{\prod_{\{x_p,\alpha\}}^{\substack{n^\mu=(0,\vec{n})\\ \vec{n}\cdot\vec{n}=1}}\delta\big(n^\mu\,A_{\alpha;\mu}(x_p)\big)\right\}\;\times \\ &\times\;\;\exp\Big\{\imath\,\mathcal{A}[\psi,\hat{D}_\mu\psi,\hat{F}]-\imath\,\mathcal{A}_S[\hat{\jmath},J_\psi,\hat{J}_{\psi\psi},\hat{\mathrm{j}}^{(\hat{F})}]\Big\}\;;\end{aligned} \tag{2.25}$$

$$\begin{aligned}\mathcal{A}[\psi,\hat{D}_\mu\psi,\hat{F}] &= \int_C d^4x_p\Big\{-\frac{1}{4}\;\hat{F}_{\alpha;\mu\nu}(x_p)\;\hat{F}^{\mu\nu}_\alpha(x_p)+ \\ &-\;\;\psi^\dagger_N(x_p)\Big[\hat{\beta}\Big(\hat{\gamma}^\mu\hat{\partial}_{p,\mu}-\imath\,\hat{\varepsilon}_p-\imath\,\hat{\gamma}^\mu\;A_{\alpha;\mu}(x_p)\;\hat{t}_\alpha+\hat{m}\Big)\Big]_{N;M}\psi_M(x_p)\Big\}\;;\end{aligned} \tag{2.26}$$

[5] In this article we neglect additional effects of symmetry breaking with the gauge field strength tensor and therefore also set the corresponding source $\hat{\mathrm{j}}^{(\hat{F})}_{\alpha;\mu\nu}(x_p)$ to zero at the final end of transformations. This normalization also allows the treatment of disordered systems with ensemble-averages over random potentials and interactions [22, 23].

$$\hat{\varepsilon}_p = \hat{\beta}\,\eta_p\,\varepsilon_+\,; \qquad \varepsilon_+ > 0\,. \tag{2.27}$$

The action $\mathcal{A}[\psi,\hat{D}_\mu\psi,\hat{F}]$ of quark matter and gauge fields has to comprise a non-hermitian $\hat{\varepsilon}_p$ matrix term (2.27) which characterizes a particular time direction in the propagation and which results into proper time ordering of quark fields and appropriate analytic convergence properties of derived Green functions.

Finally, we end this section of definitions by listing the anomalous doubling of quark fields $\Psi_M^a(x_p)$ with corresponding anti-commuting source $J_{\psi;M}^a(x_p)$ (2.28,2.29). The symmetry breaking, anomalous doubled source matrix $\hat{J}_{\psi\psi;N;M}^{b\neq a}(x_p)$ (2.31) consists of two anti-symmetric, complex, even-valued sub-matrices $\hat{j}_{\psi\psi;M;N}(x_p)$, $\hat{j}_{\psi\psi;N;M}^\dagger(x_p)$ (2.30) in the off-diagonal '12' and '21' blocks where the anti-symmetric property regards the BCS related pair condensates of fermionic quark fields. Since the gauge field strength tensor $\hat{F}_\alpha^{\mu\nu}(x_p)$ is completely anti-symmetric in its spacetime indices 'μ,ν', we have also to require this anti-symmetry for the corresponding, generating, real source matrix $\hat{\mathfrak{j}}_{\alpha;\mu\nu}^{(\hat{F})}(x_p)$ (2.32)

$$\Psi_{f,m,r}^{a(=1,2)}(x_p) = \Psi_M^{a(=1,2)}(x_p) = \Big\{\underbrace{\psi_{f,m,r}(x_p)}_{a=1}\,;\,\underbrace{\psi_{f,m,r}^*(x_p)}_{a=2}\Big\}^T = \Big\{\underbrace{\psi_M(x_p)}_{a=1}\,;\,\underbrace{\psi_M^*(x_p)}_{a=2}\Big\}^T\,; \tag{2.28}$$

$$J_{\psi;f,m,r}^{a(=1,2)}(x_p) = J_{\psi;M}^{a(=1,2)}(x_p) = \Big\{\underbrace{j_{\psi;f,m,r}(x_p)}_{a=1}\,;\,\underbrace{j_{\psi;f,m,r}^*(x_p)}_{a=2}\Big\}^T = \Big\{\underbrace{j_{\psi;M}(x_p)}_{a=1}\,;\,\underbrace{j_{\psi;M}^*(x_p)}_{a=2}\Big\}^T\,; \tag{2.29}$$

$$\hat{j}_{\psi\psi;M;N}(x_p) = -\hat{j}_{\psi\psi;M;N}^T(x_p)\,; \qquad \hat{j}_{\psi\psi;f,m,r;g,n,s}(x_p) = -\hat{j}_{\psi\psi;f,m,r;g,n,s}^T(x_p)\,; \tag{2.30}$$

$$\hat{J}_{\psi\psi;M;N}^{a\neq b}(x_p) = \begin{pmatrix} 0 & \hat{j}_{\psi\psi;M;N}(x_p) \\ \hat{j}_{\psi\psi;M;N}^\dagger(x_p) & 0 \end{pmatrix}_{M;N}^{a\neq b}\,; \tag{2.31}$$

$$\hat{\mathfrak{j}}_{\alpha;\mu\nu}^{(\hat{F})}(x_p) = -\hat{\mathfrak{j}}_{\alpha;\mu\nu}^{(\hat{F}),T}(x_p)\,; \qquad \alpha = 1,\ldots,8\,. \tag{2.32}$$

Apart from the source term $\mathcal{A}_S[\hat{\mathcal{J}},J_\psi,\hat{J}_{\psi\psi},\hat{\mathfrak{j}}^{(\hat{F})}]$ (2.22), we have also to implement the anomalous doubling of quark fields into the path integral (2.25) with the action $\mathcal{A}[\psi,\hat{D}_\mu\psi,\hat{F}]$ (2.26) in order to incorporate BCS pair condensates in the time contour propagation. This has to be achieved by Hubbard-Stratonovich transformations (HST) to self-energies which include the anomalous doubling for the fermionic matter fields. Eventually, remaining Gaussian integrals of doubled quark fields introduce the Fermi determinant with anomalous doubled one-particle Hamiltonian and self-energy which allow for the coset decomposition to the effective, BCS related actions with coset matrices $\hat{T}(x_p)$.

3 HST of Fermi- and non-Abelian gauge fields to self-energies

3.1 HST to the self-energy of the non-Abelian gauge field strength tensor

We introduce the real self-energy matrix $\hat{\mathfrak{S}}_{\alpha;\mu\nu}^{(\hat{F})}(x_p)$ with corresponding anti-symmetry as of the gauge field strength tensor $\hat{F}_\alpha^{\mu\nu}(x_p)$ in the two spacetime indices 'μ,ν' for each of the eight $\mathrm{SU}_c(N_c=3)$ colour generators. In general, HST's are related to Gaussian integrals; therefore, we start out from the Gaussian identity (3.1) with 'flat', Euclidean integration measure of the real self-energy matrix $\hat{\mathfrak{S}}_{\alpha;\mu\nu}^{(\hat{F})}(x_p)$ which couples linearly to the field strength tensor $\hat{F}_\alpha^{\mu\nu}(x_p)$ and the real, anti-symmetric source field $\hat{\mathfrak{j}}_{\alpha;\mu\nu}^{(\hat{F})}(x_p)$ in the action of the exponential

$$\begin{aligned} 1 \equiv& \int d[\hat{\mathfrak{S}}_{\alpha;\mu\nu}^{(\hat{F})}(x_p)]\;\exp\Big\{\frac{\imath}{4}\int_C d^4x_p\Big(\hat{\mathfrak{S}}_\alpha^{(\hat{F})\mu\nu}(x_p) - \hat{F}_\alpha^{\mu\nu}(x_p) - 2\,\hat{\mathfrak{j}}_\alpha^{(\hat{F})\mu\nu}(x_p)\Big)\times \\ &\times\;\Big(\hat{\mathfrak{S}}_{\alpha;\mu\nu}^{(\hat{F})}(x_p) - \hat{F}_{\alpha;\mu\nu}(x_p) - 2\,\hat{\mathfrak{j}}_{\alpha;\mu\nu}^{(\hat{F})}(x_p)\Big)\Big\}\,. \end{aligned} \tag{3.1}$$

The Gaussian integral (3.1) consists of the self-energy matrix $\hat{\mathfrak{S}}_{\alpha;\mu\nu}^{(\hat{F})}(x_p)$ which is shifted by $\hat{F}_{\alpha;\mu\nu}(x_p)$ and by the source $\hat{\mathfrak{j}}_{\alpha;\mu\nu}^{(\hat{F})}(x_p)$. In consequence, one obtains the standard relation of the HST where the quadratic term of the field strength

tensor $\hat{F}_{\alpha;\mu\nu}(x_p)$ is reduced to a linear coupling with the self-energy in a 'Euclidean' Gaussian integral. The quadratic term of the field strength $\hat{F}_{\alpha;\mu\nu}(x_p)$ on the left-hand side of (3.2) is modified by a linear coupling to the symmetry-breaking source field $\hat{\mathfrak{j}}_{\alpha;\mu\nu}^{(\hat{F})}(x_p)$ so that we also consider the corresponding part of the source action $\mathcal{A}_S$ (2.22)

$$\begin{aligned}
&\exp\Big\{-\imath\int_C d^4x_p\;\Big(\frac{1}{4}\;\hat{F}_\alpha^{\mu\nu}(x_p)\;\hat{F}_{\alpha;\mu\nu}(x_p)+\hat{\mathfrak{j}}_{\alpha;\mu\nu}^{(\hat{F})}(x_p)\;\hat{F}_\alpha^{\mu\nu}(x_p)\Big)\Big\}=\int d[\hat{\mathfrak{S}}_{\alpha;\mu\nu}^{(\hat{F})}(x_p)]\;\times \qquad (3.2)\\
&\times\;\exp\Big\{\imath\int_C d^4x_p\Big(\frac{1}{4}\;\hat{\mathfrak{S}}_\alpha^{(\hat{F})\mu\nu}(x_p)\;\hat{\mathfrak{S}}_{\alpha;\mu\nu}^{(\hat{F})}(x_p)-\frac{1}{2}\Big(\hat{F}_{\alpha;\mu\nu}(x_p)+2\;\hat{\mathfrak{j}}_{\alpha;\mu\nu}^{(\hat{F})}(x_p)\Big)\;\hat{\mathfrak{S}}_\alpha^{(\hat{F})\mu\nu}(x_p)+\\
&+\;\hat{\mathfrak{j}}_{\alpha;\mu\nu}^{(\hat{F})}(x_p)\;\hat{\mathfrak{j}}_\alpha^{(\hat{F})\mu\nu}(x_p)\Big)\Big\}\;.
\end{aligned}$$

In a subsequent step (3.3), one replaces the field strength tensor $\hat{F}_{\alpha;\mu\nu}(x_p)$ in its linear coupling to the self-energy $\hat{\mathfrak{S}}_\alpha^{(\hat{F})\mu\nu}(x_p)$ by the original expression with the gauge field $A_{\alpha;\mu}(x_p)$ (2.5,2.6). Consequently, one gains the actions in the last two lines of (3.3) which combine the self-energy $\hat{\mathfrak{S}}_\alpha^{(\hat{F})\mu\nu}(x_p)$ and its spacetime derivative to a quadratic and linear part of the gauge field $A_{\alpha;\mu}(x_p)$, respectively. The resulting, quadratic and linear term of $A_{\alpha;\mu}(x_p)$ can be removed by Gaussian integration in later steps of the derivation. We have also to include a non-hermitian $\hat{\mathfrak{e}}_p^{(\hat{F})}$ matrix term (3.4) for appropriate time-ordering and convergence properties of contour time Green functions (similar to $\hat{\varepsilon}_p$ (2.27) for the propagation of quark fields)

$$\begin{aligned}
&\exp\Big\{-\frac{\imath}{2}\int_C d^4x_p\;\hat{F}_{\alpha;\mu\nu}(x_p)\;\hat{\mathfrak{S}}_\alpha^{(\hat{F})\mu\nu}(x_p)\Big\}= \qquad (3.3)\\
&=\;\exp\Big\{-\frac{\imath}{2}\int_C d^4x_p\;\Big(\hat{\partial}_{p,\mu}A_{\alpha;\nu}(x_p)-\hat{\partial}_{p,\nu}A_{\alpha;\mu}(x_p)+C_{\alpha\beta\gamma}\;A_{\beta;\mu}(x_p)\;A_{\gamma;\nu}(x_p)\Big)\;\hat{\mathfrak{S}}_\alpha^{(\hat{F})\mu\nu}(x_p)\Big\}=\\
&=\;\exp\Big\{-\frac{\imath}{2}\int_C d^4x_p\;A_{\beta;\mu}(x_p)\;\Big[-\imath\;\hat{\mathfrak{e}}_p^{(\hat{F})}+C_{\alpha\beta\gamma}\;\hat{\mathfrak{S}}_\alpha^{(\hat{F})\mu\nu}(x_p)\Big]\;A_{\gamma;\nu}(x_p)\Big\}\times\\
&\times\;\exp\Big\{-\frac{\imath}{2}\int_C d^4x_p\Big(-A_{\alpha;\nu}(x_p)\;\big(\hat{\partial}_{p,\mu}\hat{\mathfrak{S}}_\alpha^{(\hat{F})\mu\nu}(x_p)\big)+A_{\alpha;\mu}(x_p)\;\big(\hat{\partial}_{p,\nu}\hat{\mathfrak{S}}_\alpha^{(\hat{F})\mu\nu}(x_p)\big)\Big)\Big\}=\\
&=\;\exp\Big\{-\frac{\imath}{2}\int_C d^4x_p\;A_{\beta;\mu}(x_p)\;\Big[-\imath\;\hat{\mathfrak{e}}_p^{(\hat{F})}+C_{\alpha\beta\gamma}\;\hat{\mathfrak{S}}_\alpha^{(\hat{F})\mu\nu}(x_p)\Big]\;A_{\gamma;\nu}(x_p)\Big\}\times\\
&\times\;\exp\Big\{-\imath\int_C d^4x_p\;A_{\alpha;\mu}(x_p)\;\big(\hat{\partial}_{p,\nu}\hat{\mathfrak{S}}_\alpha^{(\hat{F})\mu\nu}(x_p)\big)\Big\}\;;\\
&\Big(\hat{\mathfrak{e}}_p^{(\hat{F})}\Big)_{\beta\gamma}^{\mu\nu}=\eta_p\;\mathfrak{e}_+^{(\hat{F})}\;\delta_{\beta\gamma}\;\delta^{\mu\nu}\;;\qquad\mathfrak{e}_+^{(\hat{F})}>0\;. \qquad (3.4)
\end{aligned}$$

The combination of Eqs. (3.2,3.3) leads to relation (3.5) which transforms the quadratic term of the field strength tensor and the linear coupling to the source field $\hat{\mathfrak{j}}_{\alpha;\mu\nu}^{(\hat{F})}(x_p)$ to a Euclidean Gaussian integral of the corresponding self-energy matrix $\hat{\mathfrak{S}}_\alpha^{(\hat{F})\mu\nu}(x_p)$. Apart from the linear derivative coupling to the gauge field $A_{\alpha;\mu}(x_p)$ (last line of (3.5)), the self-energy matrix is contained as a kind of '*inverse variance*' in an action with quadratic gauge fields $A_{\alpha;\mu}(x_p)$ which can be eliminated by Gaussian integration after adding the coupling to the bilinear quark fields in $\bar{\psi}(x_p)\hat{\not{D}}(x_p)\,\psi(x_p)$

$$\begin{aligned}
&\exp\Big\{-\imath\int_C d^4x_p\;\Big(\frac{1}{4}\;\hat{F}_\alpha^{\mu\nu}(x_p)\;\hat{F}_{\alpha;\mu\nu}(x_p)+\hat{\mathfrak{j}}_{\alpha;\mu\nu}^{(\hat{F})}(x_p)\;\hat{F}_\alpha^{\mu\nu}(x_p)\Big)\Big\}= \qquad (3.5)\\
&=\;\int d[\hat{\mathfrak{S}}_{\alpha;\mu\nu}^{(\hat{F})}(x_p)]\;\exp\Big\{\imath\int_C d^4x_p\;\Big(\frac{1}{4}\;\hat{\mathfrak{S}}_\alpha^{(\hat{F})\mu\nu}(x_p)\;\hat{\mathfrak{S}}_{\alpha;\mu\nu}^{(\hat{F})}(x_p)+\\
&-\;\hat{\mathfrak{j}}_{\alpha;\mu\nu}^{(\hat{F})}(x_p)\;\hat{\mathfrak{S}}_\alpha^{(\hat{F})\mu\nu}(x_p)+\hat{\mathfrak{j}}_{\alpha;\mu\nu}^{(\hat{F})}(x_p)\;\hat{\mathfrak{j}}_\alpha^{(\hat{F})\mu\nu}(x_p)\Big)\Big\}\times\\
&\times\;\exp\Big\{-\frac{\imath}{2}\int_C d^4x_p\;A_{\beta;\mu}(x_p)\;\Big[-\imath\;\hat{\mathfrak{e}}_p^{(\hat{F})}+C_{\alpha\beta\gamma}\;\hat{\mathfrak{S}}_\alpha^{(\hat{F})\mu\nu}(x_p)\Big]\;A_{\gamma;\nu}(x_p)\Big\}\times
\end{aligned}$$

$$\times \quad \exp\Big\{-\imath\int_C d^4x_p\; A_{\alpha;\mu}(x_p)\;\big(\hat{\partial}_{p,\nu}\hat{\mathfrak{S}}_{\alpha}^{(\hat{F})\mu\nu}(x_p)\big)\Big\}\,.$$

Eventually, we can insert the completed HST (3.5) for the quadratic field strength tensor with its linear source-coupling into the path integral (2.25,2.26). The delta-function of the gauge field $A_{\alpha;\mu}(x_p)$ in (2.25,2.26), caused by the axial gauge, is taken into account by the standard integral representation with auxiliary, real fields $s_\alpha(x_p)$ ($\alpha,\,\beta,\,\ldots=1,\ldots,8$). Furthermore, the resulting generating function $Z[\hat{\mathscr{J}},J_\psi,\hat{J}_{\psi\psi},\hat{\mathfrak{j}}^{(\hat{F})}]$ (3.6) is grouped into integration terms which consist of the 'gauge field strength' self-energy $d[\hat{\mathfrak{S}}_{\alpha;\mu\nu}^{(\hat{F})}(x_p)]$, the fermionic quark fields $d[\psi_M^\dagger(x_p),\,\psi_M(x_p)]$, the auxiliary, real integration variables $d[s_\alpha(x_p)]$ for the delta-function of the axial gauge, subsequently followed by remaining Gaussian integration $d[A_{\alpha;\mu}(x_p)]$ of the gauge field

$$\begin{aligned}
Z[\hat{\mathscr{J}},J_\psi,\hat{J}_{\psi\psi},\hat{\mathfrak{j}}^{(\hat{F})}] &= \int d[\hat{\mathfrak{S}}_{\alpha;\mu\nu}^{(\hat{F})}(x_p)]\;\exp\Big\{\imath\int_C d^4x_p\,\Big(\frac{1}{4}\,\hat{\mathfrak{S}}_{\alpha}^{(\hat{F})\mu\nu}(x_p)\;\hat{\mathfrak{S}}_{\alpha;\mu\nu}^{(\hat{F})}(x_p)+ \\
&- \;\hat{\mathfrak{j}}_{\alpha;\mu\nu}^{(\hat{F})}(x_p)\;\hat{\mathfrak{S}}_{\alpha}^{(\hat{F})\mu\nu}(x_p)+\hat{\mathfrak{j}}_{\alpha;\mu\nu}^{(\hat{F})}(x_p)\;\hat{\mathfrak{j}}_{\alpha}^{(\hat{F})\mu\nu}(x_p)\Big)\Big\}\;\times \\
&\times\;\int d[\psi_M^\dagger(x_p),\,\psi_M(x_p)]\;\exp\Big\{-\imath\int_C d^4x_p\;\psi_N^\dagger(x_p)\;\Big[\hat{\beta}\Big(\hat{\gamma}^\mu\;\hat{\partial}_{p,\mu}-\imath\,\hat{\varepsilon}_p+\hat{m}\Big)\Big]_{N;M}\;\psi_M(x_p)\Big\}\times \\
&\times\;\exp\Big\{-\frac{\imath}{2}\int_C d^4x_p\Big(J_{\psi;M}^{\dagger,a}(x_p)\;\hat{S}^{ab}\;\Psi_M^b(x_p)+\Psi_M^{\dagger,a}(x_p)\;\hat{S}^{ab}\;J_{\psi;M}^b(x_p)\Big)\Big\} \\
&\times\;\exp\Big\{-\frac{\imath}{2}\int_C d^4x_p\;\Psi_N^{\dagger,b}(x_p)\;\hat{J}_{\psi\psi;N;M}^{b\neq a}(x_p)\;\Psi_M^a(x_p)\Big\} \\
&\times\;\exp\Big\{-\frac{\imath}{2}\int_C d^4x_p\;d^4y_q\;\Psi_N^{\dagger,b}(y_q)\;\hat{\mathscr{J}}_{N;M}^{ba}(y_q,x_p)\;\Psi_M^a(x_p)\Big\}\;\times\;\int d[s_\alpha(x_p)]\int d[A_{\alpha;\mu}(x_p)]\;\times \\
&\times\;\exp\Big\{-\frac{\imath}{2}\int_C d^4x_p\;A_{\beta;\mu}(x_p)\;\Big[-\imath\,\hat{\mathfrak{e}}_p^{(\hat{F})}+C_{\alpha\beta\gamma}\;\hat{\mathfrak{S}}_{\alpha}^{(\hat{F})\mu\nu}(x_p)\Big]\;A_{\gamma;\nu}(x_p)\Big\}\times \\
&\times\;\exp\Big\{-\imath\int_C d^4x_p\;A_{\beta;\mu}(x_p)\;\Big[\big(\hat{\partial}_{p,\kappa}\hat{\mathfrak{S}}_{\beta}^{(\hat{F})\mu\kappa}(x_p)\big)-\psi_{g,\overline{N}}^\dagger(x_p)\;\big[\hat{\beta}\big(\imath\;\hat{\gamma}^\mu\;\hat{t}_\beta\big)\big]_{\overline{N};\overline{M}}\,\psi_{f,\overline{M}}(x_p)\;\delta_{g,f}-s_\beta(x_p)\;n^\mu\Big]\Big\}.
\end{aligned}\tag{3.6}$$

We split the Gaussian integration part (3.7) of gauge fields $A_{\alpha;\mu}(x_p)$ from the generating function in (3.6) and separately list the result of the integration. Aside from the inverse square root of the self-energy matrix for the gauge field strength tensor, an action appears that is quadratic in spacetime derivatives of the self-energy matrix $(\,\hat{\partial}_p^\kappa\hat{\mathfrak{S}}_{\beta;\mu\kappa}^{(\hat{F})}(x_p)\,)$, quartic in the interaction of matter fields and again quadratic in the auxiliary, real field $s_\beta(x_p)$ for axial gauge fixing. The latter action in (3.7) is locally weighted by the inverse of the field strength self-energy matrix as a kind of '*variance*', indicating the self-interaction of the gauge fields. Since the self-interaction of the gauge fields consists of a quadratic derivative, (symbolically abbreviated by $(\hat{\partial}\hat{\mathfrak{S}}^{(\hat{F})})\,(\hat{\mathfrak{S}}^{(\hat{F})})^{-\mathbf{1}}\,(\hat{\partial}\hat{\mathfrak{S}}^{(\hat{F})})$), one immediately concludes for the asymptotic freedom at high energies according to the spacetime derivative $(\,\hat{\partial}_p^\kappa\hat{\mathfrak{S}}_{\beta;\mu\kappa}^{(\hat{F})}(x_p)\,)$ which also occurs in the coupling to the quark fields

$$\begin{aligned}
&\int d[A_{\alpha;\mu}(x_p)]\;\exp\Big\{-\frac{\imath}{2}\int_C d^4x_p\;A_{\beta;\mu}(x_p)\;\Big[-\imath\,\hat{\mathfrak{e}}_p^{(\hat{F})}+C_{\alpha\beta\gamma}\;\hat{\mathfrak{S}}_{\alpha}^{(\hat{F})\mu\nu}(x_p)\Big]\;A_{\gamma;\nu}(x_p)\Big\}\times \\
&\quad\times\;\exp\Big\{-\imath\int_C d^4x_p\;A_{\beta;\mu}(x_p)\;\Big[\big(\hat{\partial}_{p,\kappa}\hat{\mathfrak{S}}_{\beta}^{(\hat{F})\mu\kappa}(x_p)\big)-\psi_N^\dagger(x_p)\;\big[\hat{\beta}\big(\imath\;\hat{\gamma}^\mu\;\hat{t}_\beta\big)\big]_{N;M}\psi_M(x_p)-s_\beta(x_p)\;n^\mu\Big]\Big\}= \\
&\quad=\;\Big\{\det\Big[\Big(-\imath\,\hat{\mathfrak{e}}_p^{(\hat{F})}+C_{\alpha\beta\gamma}\;\hat{\mathfrak{S}}_{\alpha}^{(\hat{F})\mu\nu}(x_p)\Big)_{\beta\gamma}^{\mu\nu}\Big]\Big\}^{-\mathbf{1/2}}\;\times \\
&\quad\times\;\exp\Big\{\frac{\imath}{2}\int_C d^4x_p\Big[\big(\hat{\partial}_p^\lambda\hat{\mathfrak{S}}_{\gamma;\nu\lambda}^{(\hat{F})}(x_p)\big)-\psi_{g,\overline{N}}^\dagger(x_p)\;\big[\hat{\beta}\big(\imath\;\hat{\gamma}_\nu\;\hat{t}_\gamma\big)\big]_{g,\overline{N};g',\overline{N}'}\;\delta_{g,g'}\;\psi_{g',\overline{N}'}(x_p)-s_\gamma(x_p)\;n_\nu\Big]\times \\
&\quad\times\;\Big[-\imath\,\hat{\mathfrak{e}}_p^{(\hat{F})}+C_{\alpha\beta'\gamma'}\;\hat{\mathfrak{S}}_{\alpha}^{(\hat{F})\mu'\nu'}(x_p)\Big]_{\gamma\beta}^{-\mathbf{1};\nu\mu}\;\times
\end{aligned}\tag{3.7}$$

$$\times \quad \Big[\big(\hat{\partial}_p^{\kappa}\hat{\mathfrak{S}}_{\beta;\mu\kappa}^{(\hat{F})}(x_p)\big)-\psi_{f,\overline{M}}^{\dagger}(x_p)\;\big[\hat{\beta}(\imath\,\hat{\gamma}_{\mu}\;\hat{t}_{\beta})\big]_{f,\overline{M};f',\overline{M}'}\;\delta_{f,f'}\;\psi_{f',\overline{M}'}(x_p)-s_{\beta}(x_p)\;n_{\mu}\Big]\bigg\}\;.$$

We substitute the entire Gaussian integration part (3.7) of the gauge fields $A_{\alpha;\mu}(x_p)$ into (3.6) and additionally perform the anomalous doubling of the fermionic quark fields $\psi_M^{\dagger}\psi_M=\frac{1}{2}(\Psi_M^{\dagger,b}\;\hat{S}^{ba}\;\Psi_M^{a})$ (2.20,2.21) and also include the anomalous doubling with the transpose of the one-particle Hamiltonian $\hat{\beta}\,(\hat{\gamma}^{\mu}\hat{\partial}_{p,\mu}-\imath\,\hat{\varepsilon}_p+\hat{m})_{N;M}$. The anomalous doubled path integral is listed in detail in Eq. (3.8) where the doubled, fermionic fields $\Psi_M^a(x_p)$ replace the original quark fields $\psi_M(x_p)$ in (3.6) with suitable anomalous doubling of one-particle Hamiltonian and of the quartic interaction term. However, the integration variables of the self-energy $\hat{\mathfrak{S}}_{\alpha;\mu\nu}^{(\hat{F})}(x_p)$ of the field strength tensor and of the auxiliary, real field $s_{\alpha}(x_p)$, both related to propagation of the gauge terms, are separated as background fields from the fermionic quark field integrations $d[\psi_M^{\dagger}(x_p),\,\psi_M(x_p)]$ in (3.8) with additional averaging $\langle\ldots\rangle_{\hat{\mathfrak{S}}^{(\hat{F})},s_{\alpha}}^{\mathbf{Eq.(3.9)}}$ (3.9) of a 'background generating function'

$$\begin{aligned}
Z[\hat{\jmath},J_{\psi},\hat{J}_{\psi\psi},\hat{\mathfrak{j}}^{(\hat{F})}] &= \bigg\langle \boldsymbol{Z}\Big[\boldsymbol{\hat{\mathfrak{S}}^{(\hat{F})},s_{\alpha};\hat{\mathfrak{j}}^{(\hat{F})};\mathbf{Eq.\;(3.9)}}\Big]\times\int d[\psi_M^{\dagger}(x_p),\,\psi_M(x_p)]\;\times\exp\bigg\{-\frac{\imath}{2}\int_C d^4x_p\;\times \qquad (3.8)\\
&\times\quad \Psi_N^{\dagger,b}(x_p)\begin{pmatrix}[\hat{\beta}(\hat{\gamma}^{\mu}\hat{\partial}_{p,\mu}-\imath\,\hat{\varepsilon}_p+\hat{m})]_{N;M} & \hat{\jmath}_{\psi\psi;N;M}(x_p)\\ \hat{\jmath}_{\psi\psi;N;M}^{\dagger}(x_p) & -[\hat{\beta}(\hat{\gamma}^{\mu}\hat{\partial}_{p,\mu}-\imath\,\hat{\varepsilon}_p+\hat{m})]_{N;M}^T\end{pmatrix}_{N;M}^{ba}\Psi_M^a(x_p)\bigg\}\times\\
&\times\quad\exp\bigg\{-\frac{\imath}{2}\int_C d^4x_p\;d^4y_q\;\Psi_N^{\dagger,b}(y_q)\;\hat{\jmath}_{N;M}^{ba}(y_q,x_p)\;\Psi_M^a(x_p)\bigg\}\\
&\times\quad\exp\bigg\{-\frac{\imath}{2}\int_C d^4x_p\Big(J_{\psi;M}^{\dagger,a}(x_p)\;\hat{S}^{ab}\;\Psi_M^b(x_p)+\Psi_M^{\dagger,a}(x_p)\;\hat{S}^{ab}\;J_{\psi;M}^b(x_p)\Big)\bigg\}\\
&\times\quad\exp\bigg\{-\frac{\imath}{2}\int_C d^4x_p\Big[\big(\hat{\partial}_p^{\lambda}\hat{\mathfrak{S}}_{\gamma;\nu\lambda}^{(\hat{F})}(x_p)\big)-s_{\gamma}(x_p)\;n_{\nu}\Big]\Big[-\imath\;\hat{\mathfrak{e}}_p^{(\hat{F})}+C_{\alpha\beta'\gamma'}\;\hat{\mathfrak{S}}_{\alpha}^{(\hat{F})\mu'\nu'}(x_p)\Big]_{\gamma\beta}^{\mathbf{-1};\nu\mu}\times\\
&\times\quad\Psi_{g,\overline{N}}^{\dagger,b}(x_p)\begin{pmatrix}[\hat{\beta}(\imath\,\hat{\gamma}_{\mu}\;\hat{t}_{\beta})]_{g,\overline{N};f,\overline{M}} & 0\\ 0 & -[\hat{\beta}(\imath\,\hat{\gamma}_{\mu}\;\hat{t}_{\beta})]_{g,\overline{N};f,\overline{M}}^T\end{pmatrix}_{g,\overline{N};f,\overline{M}}^{ba}\delta_{g,f}\;\Psi_M^a(x_p)\bigg\}\times\\
&\times\quad\exp\bigg\{\frac{\imath}{8}\int_C d^4x_p\;\Psi_{g',\overline{N}'}^{\dagger,b'}(x_p)\begin{pmatrix}[\hat{\beta}(\imath\,\hat{\gamma}_{\nu}\;\hat{t}_{\gamma})]_{g',\overline{N}';g,\overline{N}} & 0\\ 0 & -[\hat{\beta}(\imath\,\hat{\gamma}_{\nu}\;\hat{t}_{\gamma})]_{g',\overline{N}';g,\overline{N}}^T\end{pmatrix}_{g',\overline{N}';g,\overline{N}}^{b'b}\delta_{g',g}\;\Psi_{g,\overline{N}}^b(x_p)\;\times\\
&\times\quad\Big[-\imath\;\hat{\mathfrak{e}}_p^{(\hat{F})}+C_{\alpha\beta'\gamma'}\;\hat{\mathfrak{S}}_{\alpha}^{(\hat{F})\mu'\nu'}(x_p)\Big]_{\gamma\beta}^{\mathbf{-1};\nu\mu}\;\times\\
&\times\quad\Psi_{f',\overline{M}'}^{\dagger,a'}(x_p)\begin{pmatrix}[\hat{\beta}(\imath\,\hat{\gamma}_{\mu}\;\hat{t}_{\beta})]_{f',\overline{M}';f,\overline{M}} & 0\\ 0 & -[\hat{\beta}(\imath\,\hat{\gamma}_{\mu}\;\hat{t}_{\beta})]_{f',\overline{M}';f,\overline{M}}^T\end{pmatrix}_{f',\overline{M}';f,\overline{M}}^{a'a}\delta_{f',f}\;\Psi_{f,\overline{M}}^a(x_p)\bigg\}\bigg\rangle\;;
\end{aligned}$$

$$\begin{aligned}
&\bigg\langle\boldsymbol{Z}\Big[\boldsymbol{\hat{\mathfrak{S}}^{(\hat{F})},s_{\alpha};\hat{\mathfrak{j}}^{(\hat{F})};\mathbf{Eq.\;(3.9)}}\Big]\Big(\mathbf{fields}\Big)\bigg\rangle= \qquad (3.9)\\
&=\quad\bigg\langle\int d[\hat{\mathfrak{S}}_{\alpha;\mu\nu}^{(\hat{F})}(x_p)]\;d[s_{\alpha}(x_p)]\;\exp\bigg\{\imath\int_C d^4x_p\;\bigg(\frac{1}{4}\;\hat{\mathfrak{S}}_{\alpha}^{(\hat{F})\mu\nu}(x_p)\;\hat{\mathfrak{S}}_{\alpha;\mu\nu}^{(\hat{F})}(x_p)-\hat{\mathfrak{j}}_{\alpha;\mu\nu}^{(\hat{F})}(x_p)\;\hat{\mathfrak{S}}_{\alpha}^{(\hat{F})\mu\nu}(x_p)\;+\\
&+\quad\hat{\mathfrak{j}}_{\alpha;\mu\nu}^{(\hat{F})}(x_p)\;\hat{\mathfrak{j}}_{\alpha}^{(\hat{F})\mu\nu}(x_p)\bigg)\bigg\}\times\bigg\{\det\Big[\Big(-\imath\;\hat{\mathfrak{e}}_p^{(\hat{F})}+C_{\alpha\beta\gamma}\;\hat{\mathfrak{S}}_{\alpha}^{(\hat{F})\mu\nu}(x_p)\Big)_{\beta\gamma}^{\mu\nu}\Big]\bigg\}^{\mathbf{-1/2}}\times\\
&\times\quad\exp\bigg\{\frac{\imath}{2}\int_C d^4x_p\Big[\big(\hat{\partial}_p^{\lambda}\hat{\mathfrak{S}}_{\gamma;\nu\lambda}^{(\hat{F})}(x_p)\big)-s_{\gamma}(x_p)\;n_{\nu}\Big]\times\\
&\times\quad\Big[-\imath\;\hat{\mathfrak{e}}_p^{(\hat{F})}+C_{\alpha\beta'\gamma'}\;\hat{\mathfrak{S}}_{\alpha}^{(\hat{F})\mu'\nu'}(x_p)\Big]_{\gamma\beta}^{\mathbf{-1};\nu\mu}\;\Big[\big(\hat{\partial}_p^{\kappa}\hat{\mathfrak{S}}_{\beta;\mu\kappa}^{(\hat{F})}(x_p)\big)-s_{\beta}(x_p)\;n_{\mu}\Big]\bigg\}\;\times\;\Big(\mathbf{fields}\Big)\bigg\rangle\;.
\end{aligned}$$

Despite complicated and lengthy appearance, we have moved three and four point vertices of gauge fields $A_{\alpha;\mu}(x_p)$ to a background path integral (3.9) of the field strength self-energy $\hat{\mathfrak{S}}_{\alpha;\mu\nu}^{(\hat{F})}(x_p)$. Therefore, it suffices to concentrate onto the anomalous doubled quark fields (3.8) whose quartic interaction contains as two-body potential the inverse of the gauge field self-energy matrix within a completely local spacetime relation (last three lines of (3.8)).

3.2 HST to the anomalous doubled self-energy of Fermi fields

The HST of Fermi fields is not only more involved by the anomalous doubling, but also because of the 'inverse' self-energy of the field strength tensor in the quartic interaction (3.8). We recognize that the combination $C_{\alpha\beta\gamma}\;\hat{\mathfrak{S}}_{\alpha}^{(\hat{F})\mu\nu}(x_p)$ of anti-symmetric $\mathrm{SU}_c(N_c=3)$ structure constants $C_{\alpha\beta\gamma}$ and anti-symmetric spacetime indices in $\hat{\mathfrak{S}}_{\alpha}^{(\hat{F})\mu\nu}(x_p)$ defines a totally real symmetric matrix with doubled indices of $C_{\alpha\beta\gamma}\;\hat{\mathfrak{S}}_{\alpha}^{(\hat{F})\mu\nu}(x_p)=[\mathrm{Matrix}]_{\beta\gamma}^{\mu\nu}=[\mathrm{Matrix}]_{\gamma\beta}^{\nu\mu}$. Therefore, we introduce a real, orthogonal diagonalization (3.10) of $C_{\alpha\beta\gamma}\;\hat{\mathfrak{S}}_{\alpha}^{(\hat{F})\mu\nu}(x_p)$ with orthogonal matrices $\hat{\mathfrak{B}}_{\hat{F};\beta(\alpha)}^{\mu(\kappa)}(x_p)$, $\hat{\mathfrak{B}}_{\hat{F};(\alpha)\gamma}^{T,(\kappa)\nu}(x_p)=\hat{\mathfrak{B}}_{\hat{F};(\alpha)\gamma}^{\mathbf{-1},(\kappa)\nu}(x_p)$ and real eigenvalues $\hat{\mathfrak{b}}_{(\alpha;\kappa)}^{(\hat{F})}(x_p)$ [6]. This induces a change of integration measure from the 'flat', Euclidean self-energy of gauge fields $d[\hat{\mathfrak{S}}_{\alpha;\mu\nu}^{(\hat{F})}(x_p)]$ to that of $\mathrm{SO}(\,(N_c^2-1=8)\times(3+1)\,)=\mathrm{SO}(32)$. Since the self-energy $\hat{\mathfrak{S}}_{\alpha;\mu\nu}^{(\hat{F})}(x_p)$ with its diagonalized form (3.10) only occurs as a background field and in a saddle point approximation, we do not specify details of the SO(32) integration measure which has to be incorporated by a delta-function (3.12) into the 'flat', Euclidean integration degrees of freedom $d[\hat{\mathfrak{S}}_{\alpha;\mu\nu}^{(\hat{F})}(x_p)]$

$$C_{\alpha\beta\gamma}\;\hat{\mathfrak{S}}_{\alpha}^{(\hat{F})\mu\nu}(x_p) \;:=\; \sum_{(\alpha)=1,..,8}^{(\kappa)=0,..,3}\hat{\mathfrak{B}}_{\hat{F};\beta(\alpha)}^{\mu(\kappa)}(x_p)\;\hat{\mathfrak{b}}_{(\alpha;\kappa)}^{(\hat{F})}(x_p)\;\hat{\mathfrak{B}}_{\hat{F};(\alpha)\gamma}^{T,(\kappa)\nu}(x_p)\;; \tag{3.10}$$

$$d[\hat{\mathfrak{S}}_{\alpha}^{(\hat{F})\mu\nu}(x_p)] \;\to\; d[\hat{\mathfrak{S}}_{\alpha}^{(\hat{F})\mu\nu}(x_p);\hat{\mathfrak{B}}_{\hat{F};\beta(\alpha)}^{\mu(\kappa)}(x_p);\hat{\mathfrak{b}}_{(\alpha;\kappa)}^{(\hat{F})}(x_p)]\;; \tag{3.11}$$

$$\begin{aligned} d[\hat{\mathfrak{S}}_{\alpha}^{(\hat{F})\mu\nu}(x_p);\hat{\mathfrak{B}}_{\hat{F};\beta(\alpha)}^{\mu(\kappa)}(x_p);\hat{\mathfrak{b}}_{(\alpha;\kappa)}^{(\hat{F})}(x_p)] &= d[\hat{\mathfrak{S}}_{\alpha}^{(\hat{F})\mu\nu}(x_p)]\;\; d[\hat{\mathfrak{B}}_{\hat{F};\beta(\alpha)}^{\mu(\kappa)}(x_p);\hat{\mathfrak{b}}_{(\alpha;\kappa)}^{(\hat{F})}(x_p)]\times \\ &\times\;\;\Bigg\{\prod_{\{x_p\}}\delta\Bigg(C_{\alpha\beta\gamma}\;\hat{\mathfrak{S}}_{\alpha}^{(\hat{F})\mu\nu}(x_p)-\sum_{(\alpha)=1,..,8}^{(\kappa)=0,..,3}\hat{\mathfrak{B}}_{\hat{F};\beta(\alpha)}^{\mu(\kappa)}(x_p)\;\hat{\mathfrak{b}}_{(\alpha;\kappa)}^{(\hat{F})}(x_p)\;\hat{\mathfrak{B}}_{\hat{F};(\alpha)\gamma}^{T,(\kappa)\nu}(x_p)\Bigg)\Bigg\}\,. \end{aligned} \tag{3.12}$$

Moreover, the anomalous doubled gamma- and colour-matrices $\hat{\beta}\,\hat{\gamma}^{\mu}$, $\hat{t}_{\alpha}$ in (3.8) are abbreviated by the matrix symbols $\hat{\mathbf{\Gamma}}_{\beta;\mu;M';M}^{aa}$ and $\hat{\mathbf{\Gamma}}_{\gamma;\nu;N';N}^{bb}$ in order to simplify notation (3.13-3.15). These anomalous doubled matrices are hermitian (3.15) and are also diagonal in the isospin- (flavour-) degrees of freedom which may be additionally considered by the separate indexing $\{M';M\}\to\delta_{f',f}\;\{f',\overline{M}';f,\overline{M}\}$ according to the definitions in section 2.1

$$\hat{\mathbf{\Gamma}}_{\beta;\mu;M';M}^{aa} = \begin{pmatrix} [\hat{\beta}(\imath\,\hat{\gamma}_{\mu}\;\hat{t}_{\beta})]_{M';M} & 0 \\ 0 & [\hat{\beta}(\imath\,\hat{\gamma}_{\mu}\;\hat{t}_{\beta})]_{M';M}^{T}\end{pmatrix}^{aa}\;; \tag{3.13}$$

$$\hat{\mathbf{\Gamma}}_{\gamma;\nu;N';N}^{bb} = \begin{pmatrix} [\hat{\beta}(\imath\,\hat{\gamma}_{\nu}\;\hat{t}_{\gamma})]_{N';N} & 0 \\ 0 & [\hat{\beta}(\imath\,\hat{\gamma}_{\nu}\;\hat{t}_{\gamma})]_{N';N}^{T}\end{pmatrix}^{bb}\;; \tag{3.14}$$

$$[\hat{\beta}(\imath\,\hat{\gamma}_{\mu}\;\hat{t}_{\beta})]_{M';M}^{\dagger} = [\hat{\beta}(\imath\,\hat{\gamma}_{\mu}\;\hat{t}_{\beta})]_{M';M}\;;\quad [\hat{\beta}(\imath\,\hat{\gamma}_{\mu}\;\hat{t}_{\beta})]_{M';M}=\delta_{f',f}\;[\hat{\beta}(\imath\,\hat{\gamma}_{\mu}\;\hat{t}_{\beta})]_{f',\overline{M}';f,\overline{M}}\,. \tag{3.15}$$

Proceeding with the diagonalization (3.10-3.12) and abbreviations (3.13-3.15), it is possible to reduce the anomalous doubled, quartic interaction of Fermi fields in (3.8) to relation (3.16). According to the diagonalization of (3.10), one

[6] Although we have selected similar indices '$\alpha=1,\ldots,8$' and '$\kappa=0,1,2,3$' in anticipation of the number of colour generators $\hat{t}_{\alpha}$ and the labeling of spacetime, this summation over 32 degrees of freedom can be realized by arbitrary indexing, as far as only 32 independent additions with the eigenvalues $\hat{\mathfrak{b}}_{(\alpha;\kappa)}^{(\hat{F})}(x_p)$ and eigenvectors in $\hat{\mathfrak{B}}_{\hat{F};\beta(\alpha)}^{\mu(\kappa)}(x_p)$, $\hat{\mathfrak{B}}_{\hat{F};(\alpha)\gamma}^{T,(\kappa)\nu}(x_p)$ are concerned; therefore, we need not distinguish between contravariant and covariant spacetime indexing and set the introduced indices '$(\alpha)=1,\ldots,8$' and '$(\kappa)=0,1,2,3$' into parentheses in order to emphasize their difference to the other labeling for colour generators and spacetime degrees of freedom with metric tensor $\hat{\eta}^{\mu\nu}$!

achieves the sum of $(N_c^2-1=8)\times(3+1)=32$ interaction terms of anomalous doubled quark fields, each with a different diagonalized 'two-body potential' $[-\imath\,\hat{\mathfrak{e}}_p^{(\hat{F})}+\hat{\mathfrak{b}}_{(\alpha;\kappa)}^{(\hat{F})}(x_p)\,]^{\mathbf{-1}}$ of the 32 eigenvalues $\hat{\mathfrak{b}}_{(\alpha;\kappa)}^{(\hat{F})}(x_p)$. Furthermore, the diagonalizing, orthogonal eigenvector matrices $\hat{\mathfrak{B}}_{\hat{F};\beta(\alpha)}^{\mu(\kappa)}(x_p)$, $\hat{\mathfrak{B}}_{\hat{F};(\alpha)\gamma}^{T,(\kappa)\nu}(x_p)$ can be shifted into scalar products of anomalous doubled Fermi fields by defining the potentials $\hat{\boldsymbol{\mathcal{V}}}_{(\alpha);M';M}^{(\hat{F})a;(\kappa)}(x_p)$ (3.17,3.18) in terms of (3.13-3.15) and (3.10)

$$\begin{aligned}
&\exp\bigg\{\frac{\imath}{8}\int_C d^4x_p\Big(\Psi_{N'}^{\dagger,b}(x_p)\,\hat{\mathbf{\Gamma}}_{\gamma;\nu;N';N}^{bb}\;\hat{S}^{bb}\;\Psi_N^b(x_p)\Big)\;\Big[-\imath\,\hat{\mathfrak{e}}_p^{(\hat{F})}+C_{\alpha\beta'\gamma'}\;\hat{\mathfrak{S}}_\alpha^{(\hat{F})\mu'\nu'}(x_p)\Big]_{\gamma\beta}^{\mathbf{-1};\nu\mu}\times &(3.16)\\
&\times\;\;\Big(\Psi_{M'}^{\dagger,a}(x_p)\,\hat{\mathbf{\Gamma}}_{\beta;\mu;M';M}^{aa}\;\hat{S}^{aa}\;\Psi_M^a(x_p)\Big)\bigg\}=\exp\bigg\{\frac{\imath}{8}\int_C d^4x_p\sum_{(\alpha)=1,..,8}^{(\kappa)=0,..,3}\Psi_{N'}^{\dagger,b}(x_p)\;\hat{\boldsymbol{\mathcal{V}}}_{(\alpha);N';N}^{(\hat{F})b;(\kappa)}(x_p)\;\hat{S}^{bb}\;\Psi_N^b(x_p)\;\times\\
&\times\;\;\Big[-\imath\,\hat{\mathfrak{e}}_p^{(\hat{F})}+\hat{\mathfrak{b}}_{(\alpha;\kappa)}^{(\hat{F})}(x_p)\Big]^{\mathbf{-1}}\;\Psi_{M'}^{\dagger,a}(x_p)\;\hat{\boldsymbol{\mathcal{V}}}_{(\alpha);M';M}^{(\hat{F})a;(\kappa)}(x_p)\;\hat{S}^{aa}\;\Psi_M^a(x_p)\bigg\}\;;
\end{aligned}$$

$$\hat{\boldsymbol{\mathcal{V}}}_{(\alpha);M';M}^{(\hat{F})a;(\kappa)}(x_p)\;=\;\hat{\mathbf{\Gamma}}_{\beta;\mu;M';M}^{aa}\;\hat{\mathfrak{B}}_{\hat{F};(\alpha)\beta}^{T;(\kappa)\mu}(x_p)=\hat{\mathbf{\Gamma}}_{\beta;\mu;f',\overline{M}';f,\overline{M}}^{aa}\;\delta_{f',f}\;\hat{\mathfrak{B}}_{\hat{F};\beta(\alpha)}^{\mu(\kappa)}(x_p)\;; \tag{3.17}$$

$$\hat{\boldsymbol{\mathcal{V}}}_{(\alpha);N';N}^{(\hat{F})b;(\kappa)}(x_p)\;=\;\hat{\mathbf{\Gamma}}_{\gamma;\nu;N';N}^{bb}\;\hat{\mathfrak{B}}_{\hat{F};\gamma(\alpha)}^{\nu(\kappa)}(x_p)=\hat{\mathbf{\Gamma}}_{\gamma;\nu;g',\overline{N}';g,\overline{N}}^{bb}\;\delta_{g',g}\;\hat{\mathfrak{B}}_{\hat{F};\gamma(\alpha)}^{\nu(\kappa)}(x_p)\;. \tag{3.18}$$

This allows to perform the dyadic product of anomalous doubled quark fields to the required density matrix $\hat{R}_{M;N'}^{ab}(x_p)$ with BCS paired terms in the off-diagonal blocks ($a\neq b$) so that the HST becomes possible with an anomalous doubled self-energy of quark matter fields in a Gaussian integral

$$\Psi_M^a(x_p)\otimes\Psi_{N'}^{\dagger,b}(x_p)=\begin{pmatrix}\psi_M(x_p)\\ \psi_M^*(x_p)\end{pmatrix}\otimes\Big(\psi_{N'}^*(x_p)\,;\,\psi_{N'}(x_p)\Big)=\hat{R}_{M;N'}^{ab}(x_p)\;. \tag{3.19}$$

Combination of the entire relations (3.10-3.19) for the path integral (3.8) yields Eq. (3.20) with density matrices (3.19) for the anomalous doubled, quartic interaction part of Fermi fields

$$\begin{aligned}
&\exp\bigg\{\frac{\imath}{8}\int_C d^4x_p\Big(\Psi_{N'}^{\dagger,b}(x_p)\,\hat{\mathbf{\Gamma}}_{\gamma;\nu;N';N}^{bb}\;\hat{S}^{bb}\;\Psi_N^b(x_p)\Big)\;\Big[-\imath\,\hat{\mathfrak{e}}_p^{(\hat{F})}+C_{\alpha\beta'\gamma'}\;\hat{\mathfrak{S}}_\alpha^{(\hat{F})\mu'\nu'}(x_p)\Big]_{\gamma\beta}^{\mathbf{-1};\nu\mu}\times &(3.20)\\
&\times\;\;\Big(\Psi_{M'}^{\dagger,a}(x_p)\,\hat{\mathbf{\Gamma}}_{\beta;\mu;M';M}^{aa}\;\hat{S}^{aa}\;\Psi_M^a(x_p)\Big)\bigg\}=\exp\bigg\{-\frac{\imath}{8}\int_C d^4x_p\sum_{(\alpha)=1,..,8}^{(\kappa)=0,..,3}\bigg(\hat{R}_{M;N'}^{ab}(x_p)\;\hat{\boldsymbol{\mathcal{V}}}_{(\alpha);N';N}^{(\hat{F})b;(\kappa)}(x_p)\;\hat{S}^{bb}\;\times\\
&\times\;\;\hat{R}_{N;M'}^{ba}(x_p)\;\hat{\boldsymbol{\mathcal{V}}}_{(\alpha);M';M}^{(\hat{F})a;(\kappa)}(x_p)\;\hat{S}^{aa}\;\Big[-\imath\,\hat{\mathfrak{e}}_p^{(\hat{F})}+\hat{\mathfrak{b}}_{(\alpha;\kappa)}^{(\hat{F})}(x_p)\Big]^{\mathbf{-1}}\bigg)\bigg\}\;.
\end{aligned}$$

However, the dyadic product (3.19) to the anomalous doubled density matrix $\hat{R}_{M;N'}^{ab}(x_p)$ does not regard the $8\times(3{+}1)=32$ different potentials $\hat{\boldsymbol{\mathcal{V}}}_{(\alpha);M';M}^{(\hat{F})a;(\kappa)}(x_p)$, $\hat{\boldsymbol{\mathcal{V}}}_{(\alpha);N';N}^{(\hat{F})b;(\kappa)}(x_p)$ (3.17,3.18) *which even depend on the anomalous indexing with* $a,b=1,2$. Therefore, we take an additional, unitary diagonalization $\hat{\mathfrak{U}}_{\hat{F};\overline{N};\overline{M}}^{(\alpha;\kappa)}(x_p)$ (3.22) of the hermitian, doubled '11' and '22' potentials (3.17,3.18) into real eigenvalues $\hat{\mathfrak{v}}_{\hat{F};\overline{N}}^{(\alpha;\kappa)}(x_p)$ (The indices '$(\alpha;\kappa)$' in (3.22) are also embraced by parentheses because a diagonalization is performed for every pair '$(\alpha;\kappa)$' of the $8\times(3+1)=32$ combinations without any additional summation of these !). Using the unitary properties of $\hat{\mathfrak{U}}_{\hat{F};\overline{N};\overline{M}}^{(\alpha;\kappa)}(x_p)$ in the '11' block, one straightforwardly derives conditions (3.23) for the transposed '22' block and determines through diagonalization with $\hat{\mathfrak{U}}_{\hat{F};\overline{N};\overline{M}}^{(\alpha;\kappa)}(x_p)$ new density matrices $\hat{\mathfrak{R}}_{M;N}^{(\alpha;\kappa)ab}(x_p)$. These are modified from $\hat{R}_{M;N}^{ab}(x_p)$ by the block diagonal unitary transformations $\hat{\mathfrak{U}}_{\hat{F};\overline{M};\overline{N}'}^{(\alpha;\kappa),aa}(x_p)$, $\hat{\mathfrak{U}}_{\hat{F};\overline{M}';\overline{N}}^{(\alpha;\kappa),bb,\dagger}(x_p)$ with eigenvalues $\hat{\mathfrak{v}}_{\hat{F};\overline{N}}^{(\alpha;\kappa)}(x_p)$ which do not depend on the anomalous doubling in contrast to the potentials $\hat{\boldsymbol{\mathcal{V}}}_{(\alpha);M';M}^{(\hat{F})a;(\kappa)}(x_p)$, $\hat{\boldsymbol{\mathcal{V}}}_{(\alpha);N';N}^{(\hat{F})b;(\kappa)}(x_p)$ (3.17,3.18)

$$\hat{\boldsymbol{\mathcal{V}}}_{(\alpha);M';M}^{(\hat{F})a;(\kappa)}(x_p)=\hat{\mathbf{\Gamma}}_{\beta;\mu;f',\overline{M}';f,\overline{M}}^{aa}\;\delta_{f',f}\;\hat{\mathfrak{B}}_{\hat{F};\beta(\alpha)}^{\mu(\kappa)}(x_p)= \tag{3.21}$$

$$= \left(\begin{array}{cc} \big[\hat{\beta}(\imath\,\hat{\gamma}_\mu\,\hat{t}_\beta\,\hat{\mathfrak{B}}^{\mu(\kappa)}_{\hat{F};\beta(\alpha)}(x_p)\,)\big]_{f',\overline{M}';f,\overline{M}} & 0 \\ 0 & \big[\hat{\beta}(\imath\,\hat{\gamma}_\mu\,\hat{t}_\beta\,\hat{\mathfrak{B}}^{\mu(\kappa)}_{\hat{F};\beta(\alpha)}(x_p)\,)\big]^T_{f',\overline{M}';f,\overline{M}} \end{array} \right)^{aa}_{f',\overline{M}';f,\overline{M}} \delta_{f',f}\;;$$

$$\big[\hat{\beta}(\imath\,\hat{\gamma}_\mu\,\hat{t}_\beta\,)\big]_{f',\overline{M}';f,\overline{M}}\;\hat{\mathfrak{B}}^{\mu(\kappa)}_{\hat{F};\beta(\alpha)}(x_p)=\hat{\mathfrak{U}}^{(\alpha;\kappa),\dagger}_{\hat{F};\overline{M}';\overline{N}}(x_p)\;\underbrace{\hat{\mathfrak{v}}^{(\alpha;\kappa)}_{\hat{F};\overline{N}}(x_p)}_{\text{real}}\;\hat{\mathfrak{U}}^{(\alpha;\kappa)}_{\hat{F};\overline{N};\overline{M}}(x_p)\;\delta_{f',f}\;; \tag{3.22}$$

$$\begin{array}{rclrcl} \hat{\mathfrak{U}}^{(\alpha;\kappa)}_{\hat{F};\overline{N};\overline{M}}(x_p) & \to & \hat{\mathfrak{U}}^{(\alpha;\kappa),ab}_{\hat{F};\overline{N};\overline{M}}(x_p)\;\delta_{ab}\;; & \hat{1}_{\overline{N};\overline{M}} & = & \hat{\mathfrak{U}}^{(\alpha;\kappa),\dagger}_{\hat{F};\overline{N};\overline{M}'}(x_p)\;\hat{\mathfrak{U}}^{(\alpha;\kappa)}_{\hat{F};\overline{M}';\overline{M}}(x_p)\;; \\ \hat{\mathfrak{U}}^{(\alpha;\kappa),11}_{\hat{F};\overline{N};\overline{M}}(x_p) & := & \hat{\mathfrak{U}}^{(\alpha;\kappa)}_{\hat{F};\overline{N};\overline{M}}(x_p)\;; & \hat{\mathfrak{U}}^{(\alpha;\kappa),22}_{\hat{F};\overline{N};\overline{M}}(x_p) & := & \hat{\mathfrak{U}}^{(\alpha;\kappa),*}_{\hat{F};\overline{N};\overline{M}}(x_p)\;; \\ \hat{\mathfrak{U}}^{(\alpha;\kappa),11,\dagger}_{\hat{F};\overline{M}';\overline{N}}(x_p) & := & \hat{\mathfrak{U}}^{(\alpha;\kappa),\dagger}_{\hat{F};\overline{M}';\overline{N}}(x_p)\;; & \hat{\mathfrak{U}}^{(\alpha;\kappa),22,\dagger}_{\hat{F};\overline{M}';\overline{N}}(x_p) & := & \hat{\mathfrak{U}}^{(\alpha;\kappa),T}_{\hat{F};\overline{N};\overline{M}}(x_p)\;; \end{array} \tag{3.23}$$

$$\hat{\boldsymbol{\mathcal{V}}}^{(\hat{F})a;(\kappa)}_{(\alpha);f',\overline{M}';f,\overline{M}}(x_p) = \hat{\mathfrak{U}}^{(\alpha;\kappa),aa,\dagger}_{\hat{F};\overline{M}';\overline{N}}(x_p)\;\hat{\mathfrak{v}}^{(\alpha;\kappa)}_{\hat{F};\overline{N}}(x_p)\;\hat{\mathfrak{U}}^{(\alpha;\kappa),aa}_{\hat{F};\overline{N};\overline{M}}(x_p)\;\delta_{f',f}\;; \tag{3.24}$$

$$\hat{\boldsymbol{\mathcal{V}}}^{(\hat{F})b;(\kappa)}_{(\alpha);g',\overline{N}';g,\overline{N}}(x_p) = \hat{\mathfrak{U}}^{(\alpha;\kappa),bb,\dagger}_{\hat{F};\overline{N}';\overline{M}}(x_p)\;\hat{\mathfrak{v}}^{(\alpha;\kappa)}_{\hat{F};\overline{M}}(x_p)\;\hat{\mathfrak{U}}^{(\alpha;\kappa),bb}_{\hat{F};\overline{M};\overline{N}}(x_p)\;\delta_{g',g}\;; \tag{3.25}$$

$$\hat{R}^{ab}_{M;N}(x_p) \;\to\; \hat{\mathfrak{R}}^{(\alpha;\kappa)ab}_{f,\overline{M};g,\overline{N}}(x_p)=\hat{\mathfrak{U}}^{(\alpha;\kappa),aa}_{\hat{F};\overline{M};\overline{N}'}(x_p)\;\;\hat{R}^{ab}_{f,\overline{N}';g,\overline{M}'}(x_p)\;\;\hat{\mathfrak{U}}^{(\alpha;\kappa),bb,\dagger}_{\hat{F};\overline{M}';\overline{N}}(x_p)\;. \tag{3.26}$$

The transformation (3.26) of the density matrices $\hat{R}^{ab}_{M;N}(x_p)$ to $\hat{\mathfrak{R}}^{(\alpha;\kappa)ab}_{f,\overline{M};g,\overline{N}}(x_p)$ with block diagonal, unitary matrices $\hat{\mathfrak{U}}^{(\alpha;\kappa),aa}_{\hat{F};\overline{M};\overline{N}'}(x_p)$, $\hat{\mathfrak{U}}^{(\alpha;\kappa),bb,\dagger}_{\hat{F};\overline{M}';\overline{N}}(x_p)$ (3.23) changes the quartic interaction term of Fermi fields in Eq. (3.20) to relation (3.27). The quartic interaction (3.27) with the trace $\mathfrak{T}^{a(=1,2)}_{N_f,\hat{\gamma}^{(\mu)}_{mn},N_c}$ over $\hat{\mathfrak{R}}^{(\alpha;\kappa)ab}_{M;N}(x_p)$ differs from (3.20) with $\hat{R}^{ab}_{M;N'}(x_p)$ by the $8\times(3+1)=32$ eigenvalues $\hat{\mathfrak{v}}^{(\alpha;\kappa)}_{\hat{F};\overline{M}}(x_p)$ instead of the anomalous indexed potentials $\hat{\boldsymbol{\mathcal{V}}}^{(\hat{F})a;(\kappa)}_{(\alpha);f',\overline{M}';f,\overline{M}}(x_p)$ (3.17,3.18,3.21-3.26). The transformation of $\hat{R}^{ab}_{M;N}(x_p)$ to $8\times(3+1)=32$ density matrices $\hat{\mathfrak{R}}^{(\alpha;\kappa)ab}_{M;N}(x_p)$ also underlines the 32 different diagonalized 'two-body' potentials $[-\imath\,\hat{\mathfrak{e}}^{(\hat{F})}_p+\hat{\mathfrak{b}}^{(\hat{F})}_{(\alpha;\kappa)}(x_p)]^{-\mathbf{1}}$ which cause nontrivial couplings in the anomalous doubled, internal space with index $a,b=1,2$ (3.20). The unitary transformation of density matrices gives rise to a change of integration measure with $\mathrm{U}(\,(N_c^2-1=8)\times(3+1))=\mathrm{U}(32)$ which is not determined in detail because it is absorbed into the background generating function of gauge fields [24]. An additional delta-function (3.28) in the measure of the field strength self-energy $d[\hat{\mathfrak{S}}^{(\hat{F})}_{\alpha;\mu\nu}(x_p)]$ guarantees the equivalence of the two quartic interactions (3.20) and (3.27) with the transformation (3.26) to the $8\times(3+1)=32$ density matrices $\hat{\mathfrak{R}}^{(\alpha;\kappa)ab}_{M;N}(x_p)$

$$\begin{aligned} &\exp\bigg\{\frac{\imath}{8}\int_C d^4x_p\Big(\Psi^{\dagger,b}_{N'}(x_p)\,\hat{\boldsymbol{\Gamma}}^{bb}_{\gamma;\nu;N';N}\;\hat{S}^{bb}\;\Psi^b_N(x_p)\Big)\Big[-\imath\,\hat{\mathfrak{e}}^{(\hat{F})}_p+C_{\alpha\beta'\gamma'}\;\hat{\mathfrak{S}}^{(\hat{F})\mu'\nu'}_{\alpha}(x_p)\Big]^{-\mathbf{1};\nu\mu}_{\gamma\beta}\times \\ &\times\;\Big(\Psi^{\dagger,a}_{M'}(x_p)\,\hat{\boldsymbol{\Gamma}}^{aa}_{\beta;\mu;M';M}\;\hat{S}^{aa}\;\Psi^a_M(x_p)\Big)\bigg\}=\exp\bigg\{-\frac{\imath}{8}\int_C d^4x_p\sum_{(\alpha)=1,..,8}^{(\kappa)=0,..,3}\Big[-\imath\,\hat{\mathfrak{e}}^{(\hat{F})}_p+\hat{\mathfrak{b}}^{(\hat{F})}_{(\alpha\;\kappa)}(x_p)\Big]^{-\mathbf{1}}\times \\ &\times\underset{N_f,\hat{\gamma}^{(\mu)}_{mn},N_c}{\overset{a(=1,2)}{\mathfrak{Tr}}}\Big[\hat{\mathfrak{R}}^{(\alpha;\kappa)ab}_{f,\overline{M};g,\overline{N}}(x_p)\;\hat{\mathfrak{v}}^{(\alpha;\kappa)}_{\hat{F};\overline{N}}(x_p)\;\hat{S}^{bb}\;\hat{\mathfrak{R}}^{(\alpha;\kappa)ba}_{g,\overline{N};f,\overline{M}}(x_p)\;\hat{\mathfrak{v}}^{(\alpha;\kappa)}_{\hat{F};\overline{M}}(x_p)\;\hat{S}^{aa}\Big]\bigg\}\;; \end{aligned} \tag{3.27}$$

$$\begin{aligned} &\textbf{Eq. (3.12)}\to d[\hat{\mathfrak{S}}^{(\hat{F})}_{\alpha;\mu\nu}(x_p)]\;\;d[\hat{\mathfrak{B}}^{\mu(\kappa)}_{\hat{F};\beta(\alpha)}(x_p);\hat{\mathfrak{b}}^{(\hat{F})}_{(\alpha;\kappa)}(x_p)]\;\;d[\hat{\mathfrak{U}}^{(\alpha;\kappa)}_{\hat{F};\overline{N};\overline{M}}(x_p);\hat{\mathfrak{v}}^{(\alpha;\kappa)}_{\hat{F};\overline{N}}(x_p)]\times \\ &\times\;\bigg\{\prod_{\{x_p\}}\delta\bigg(C_{\alpha\beta\gamma}\;\hat{\mathfrak{S}}^{(\hat{F})\mu\nu}_{\alpha}(x_p)-\sum_{(\alpha)=1,..,8}^{(\kappa)=0,..,3}\hat{\mathfrak{B}}^{\mu(\kappa)}_{\hat{F};\beta(\alpha)}(x_p)\;\hat{\mathfrak{b}}^{(\hat{F})}_{(\alpha;\kappa)}(x_p)\;\hat{\mathfrak{B}}^{T,(\kappa)\nu}_{\hat{F};(\alpha)\gamma}(x_p)\bigg)\bigg\}\times \\ &\times\;\bigg\{\prod_{\{x_p;(\alpha)=1,..,8\}}^{\{(\kappa)=0,..,3\}}\delta\bigg(\hat{\mathfrak{U}}^{(\alpha;\kappa),11,\dagger}_{\hat{F};\overline{M}';\overline{N}}(x_p)\;\hat{\mathfrak{v}}^{(\alpha;\kappa)}_{\hat{F};\overline{N}}(x_p)\;\hat{\mathfrak{U}}^{(\alpha;\kappa),11}_{\hat{F};\overline{N};\overline{M}}(x_p)-\big[\hat{\beta}\,(\imath\,\hat{\gamma}_\mu\,\hat{t}_\beta)\big]_{\overline{M}';\overline{M}}\;\hat{\mathfrak{B}}^{\mu(\kappa)}_{\hat{F};\beta(\alpha)}(x_p)\bigg)\bigg\}\;. \end{aligned} \tag{3.28}$$

Relations (3.27,3.28) of the anomalous doubled, quartic interaction of Fermi fields allow to apply $8\times(3+1)=32$ HST's by introducing 32 independent, different self-energies $\delta\hat{\Sigma}_{M;N}^{(\alpha;\kappa)ab}(x_p)$ (3.29-3.33) for the matter fields in analogy to the super-symmetric case of Ref. [11]. The anti-hermitian, anti-symmetric BCS related self-energy parts $\imath\,\delta\hat{\Sigma}_{M;N}^{(\alpha;\kappa)a\neq b}(x_p)$ are denoted by a tilde '$\widetilde{\ }$' over the total self-energy symbol $\delta\widetilde{\Sigma}_{M;N}^{(\alpha;\kappa)ab}(x_p)$ of the quark matter fields. The anti-symmetry of $\delta\widetilde{\Sigma}_{M;N}^{(\alpha;\kappa)a\neq b}(x_p)=\imath\,\delta\hat{\Sigma}_{M;N}^{(\alpha;\kappa)a\neq b}(x_p)$ regards the BCS pairing of quarks whereas the anti-hermiticity takes into account a proper coset parametrization with the appropriate number of independent field degrees of freedom; these have to coincide with the number of independent parameters for the transformations leaving the path integral (3.8,3.9) invariant

$$\delta\widetilde{\Sigma}_{M;N}^{(\alpha;\kappa)ab}(x_p) = \left(\begin{array}{cc} \delta\hat{\Sigma}_{M;N}^{(\alpha;\kappa)11}(x_p) & \imath\,\delta\hat{\Sigma}_{M;N}^{(\alpha;\kappa)12}(x_p) \\ \imath\,\delta\hat{\Sigma}_{M;N}^{(\alpha;\kappa)21}(x_p) & \delta\hat{\Sigma}_{M;N}^{(\alpha;\kappa)22}(x_p)\end{array}\right)_{M;N}^{ab}; \tag{3.29}$$

$$\delta\hat{\Sigma}_{M;N}^{(\alpha;\kappa)12}(x_p) = -\Big(\delta\hat{\Sigma}_{M;N}^{(\alpha;\kappa)12}(x_p)\Big)^T; \qquad \delta\hat{\Sigma}_{M;N}^{(\alpha;\kappa)21}(x_p) = -\Big(\delta\hat{\Sigma}_{M;N}^{(\alpha;\kappa)21}(x_p)\Big)^T; \tag{3.30}$$

$$\delta\hat{\Sigma}_{M;N}^{(\alpha;\kappa)21}(x_p) = \Big(\delta\hat{\Sigma}_{M;N}^{(\alpha;\kappa)12}(x_p)\Big)^\dagger; \tag{3.31}$$

$$\Big(\delta\hat{\Sigma}_{M;N}^{(\alpha;\kappa)11}(x_p)\Big)^\dagger = \delta\hat{\Sigma}_{M;N}^{(\alpha;\kappa)11}(x_p); \qquad \Big(\delta\hat{\Sigma}_{M;N}^{(\alpha;\kappa)22}(x_p)\Big)^\dagger = \delta\hat{\Sigma}_{M;N}^{(\alpha;\kappa)22}(x_p); \tag{3.32}$$

$$\delta\hat{\Sigma}_{M;N}^{(\alpha;\kappa)22}(x_p) = -\Big(\delta\hat{\Sigma}_{M;N}^{(\alpha;\kappa)11}(x_p)\Big)^T; \tag{3.33}$$

$$\hat{\Sigma}_{M;N}^{(\alpha;\kappa)11}(x_p) = \sigma_D^{(\alpha;\kappa)}(x_p)\,\hat{\mathfrak{v}}_{\hat{F};\overline{M}}^{(\alpha;\kappa)}(x_p)\,\delta_{M;N} + \delta\hat{\Sigma}_{M;N}^{(\alpha;\kappa)11}(x_p); \tag{3.34}$$

$$\hat{\Sigma}_{M;N}^{(\alpha;\kappa)22}(x_p) = \sigma_D^{(\alpha;\kappa)}(x_p)\,\hat{\mathfrak{v}}_{\hat{F};\overline{M}}^{(\alpha;\kappa)}(x_p)\,\delta_{M;N} + \delta\hat{\Sigma}_{M;N}^{(\alpha;\kappa)22}(x_p). \tag{3.35}$$

According to Ref. [11], we catalogue the analogous HST's in (3.36,3.37) and briefly describe how half of the original, anomalous doubled, quartic interaction of fields $\Psi_M^a(x_p)$, $\Psi_M^{\dagger,a}(x_p)$ is transformed by the off-diagonal blocks $a\neq b$ or BCS related self-energies $\delta\widetilde{\Sigma}_{M;N}^{(\alpha;\kappa)a\neq b}(x_p)=\imath\,\delta\hat{\Sigma}_{M;N}^{(\alpha;\kappa)a\neq b}(x_p)$[7] and how the other half (3.37) is transformed by including $8\times(3+1)=32$ real, scalar, diagonal self-energy fields $\sigma_D^{(\alpha;\kappa)}(x_p)$ for the quark densities (3.34,3.35). Moreover, we emphasize the anti-hermitian kind of the 32 HST's with $\imath\,\delta\hat{\Sigma}_{M;N}^{(\alpha;\kappa)12}(x_p)$, $\imath\,\delta\hat{\Sigma}_{M;N}^{(\alpha;\kappa)21}(x_p)$ so that one achieves a suitable coset decomposition with the identical number of independent parameter fields as in the invariant transformations of the path integral (3.8,3.9). Corresponding to Ref. [11], we perform the HST's of the 32 density matrices $\hat{\mathcal{R}}_{N;M}^{(\alpha;\kappa)a\neq b}(x_p)$ for half of the quartic interaction[8] by 'flat', Euclidean Gaussian integrals of 32 self-energies $\delta\hat{\Sigma}_{M;N}^{(\alpha;\kappa)a\neq b}(x_p)$ in the off-diagonal, BCS related sector of the anomalous doubled quark fields

$$\begin{aligned}
&\exp\left\{\frac{\imath}{16}\int_C d^4x_p\,\Big(\Psi_{N'}^{\dagger,b}(x_p)\,\hat{\mathbf{\Gamma}}_{\gamma;\nu;N';N}^{bb}\,\hat{S}^{bb}\,\Psi_N^b(x_p)\Big)\Big[-\imath\,\hat{\mathfrak{e}}_p^{(\hat{F})}+C_{\alpha\beta'\gamma'}\,\hat{\mathfrak{S}}_\alpha^{(\hat{F})\mu'\nu'}(x_p)\Big]_{\gamma\beta}^{-1;\nu\mu}\times\right. \\
&\times\left.\Big(\Psi_{M'}^{\dagger,a}(x_p)\,\hat{\mathbf{\Gamma}}_{\beta;\mu;M';M}^{aa}\,\hat{S}^{aa}\,\Psi_M^a(x_p)\Big)\right\} = \int d[\delta\hat{\Sigma}_{M;N}^{(\alpha;\kappa)12}(x_p);\delta\hat{\Sigma}_{M;N}^{(\alpha;\kappa)21}(x_p)]\;\times \\
&\times\;\mathfrak{P}_1\Big(\hat{\mathfrak{b}}_{(\alpha;\kappa)}^{(\hat{F})}(x_p)\Big)\,\mathfrak{Q}_1\Big(\hat{\mathfrak{v}}_{\hat{F};\overline{M}}^{(\alpha;\kappa)}(x_p)\Big)\,\exp\left\{\frac{\imath}{8}\int_C d^4x_p\sum_{(\alpha)=1,..,8}^{(\kappa)=0,..,3}\Big[-\imath\,\hat{\mathfrak{e}}_p^{(\hat{F})}+\hat{\mathfrak{b}}_{(\alpha;\kappa)}^{(\hat{F})}(x_p)\Big]\;\times\right. \\
&\times\left.\underset{N_f,\hat{\gamma}_{mn}^{(\mu)},N_c}{\overset{a(=1,2)}{\mathfrak{Tr}}}\left[\Big(\boldsymbol{\delta\hat{\Sigma}_{f,\overline{M};g,\overline{N}}^{(\alpha;\kappa)a\neq b}(x_p)}\Big)\frac{\left(\begin{array}{cc}1&\\&\mathbf{-1}\end{array}\right)^{bb}}{\hat{\mathfrak{v}}_{\hat{F};\overline{N}}^{(\alpha;\kappa)}(x_p)}\Big(\boldsymbol{\delta\hat{\Sigma}_{g,\overline{N};f,\overline{M}}^{(\alpha;\kappa)b\neq a}(x_p)}\Big)\frac{\left(\begin{array}{cc}1&\\&\mathbf{-1}\end{array}\right)^{aa}}{\hat{\mathfrak{v}}_{\hat{F};\overline{M}}^{(\alpha;\kappa)}(x_p)}\right]\right\}\times \\
&\times\;\exp\left\{\frac{\imath}{4}\int_C d^4x_p\sum_{(\alpha)=1,..,8}^{(\kappa)=0,..,3}\underset{N_f,\hat{\gamma}_{mn}^{(\mu)},N_c}{\overset{a(=1,2)}{\mathfrak{Tr}}}\left[\Big(\delta\hat{\Sigma}_{M;N}^{(\alpha;\kappa)\boldsymbol{a\neq b}}(x_p)\Big)\left(\begin{array}{cc}1&\\&-1\end{array}\right)^{bb}\Big(\hat{\mathcal{R}}_{N;M}^{(\alpha;\kappa)\boldsymbol{b\neq a}}(x_p)\Big)\left(\begin{array}{cc}1&\\&-1\end{array}\right)^{aa}\right]\right\} =
\end{aligned} \tag{3.36}$$

[7]Note the boldface typing $\boldsymbol{\imath\,\delta\hat{\Sigma}_{M;N}^{(\alpha;\kappa)12}(x_p)}$, $\boldsymbol{\imath\,\delta\hat{\Sigma}_{M;N}^{(\alpha;\kappa)21}(x_p)}$ in (3.36) !

[8]Pre-factor $\imath/16$ instead of $\imath/8$ as in (3.27) for the 'half' of the quartic interaction of Fermi fields !

$$
\begin{aligned}
&= \int d[\delta\hat{\Sigma}_{M;N}^{(\alpha;\kappa)12}(x_p);\delta\hat{\Sigma}_{M;N}^{(\alpha;\kappa)21}(x_p)] \;\; \mathfrak{P}_1\Big(\hat{\mathfrak{b}}_{(\alpha;\kappa)}^{(\hat{F})}(x_p)\Big)\;\mathfrak{Q}_1\Big(\hat{\mathfrak{v}}_{\hat{F};\overline{M}}^{(\alpha;\kappa)}(x_p)\Big)\;\times \\
&\times\;\exp\bigg\{\frac{\imath}{8}\int_C d^4x_p\sum_{(\alpha)=1,..,8}^{(\kappa)=0,..,3}\Big[-\imath\,\hat{\mathfrak{e}}_p^{(\hat{F})}+\hat{\mathfrak{b}}_{(\alpha;\kappa)}^{(\hat{F})}(x_p)\Big]\times \\
&\times\;\mathop{\mathfrak{Tr}}_{N_f,\hat{\gamma}_{mn}^{(\mu)},N_c}^{a(=1,2)}\Bigg[\Big(\imath\,\boldsymbol{\delta\hat{\Sigma}_{f,\overline{M};g,\overline{N}}^{(\alpha;\kappa)a\neq b}(x_p)}\Big)\frac{\begin{pmatrix}1&\\&\mathbf{+1}\end{pmatrix}^{bb}}{\hat{\mathfrak{v}}_{\hat{F};\overline{N}}^{(\alpha;\kappa)}(x_p)}\Big(\imath\,\boldsymbol{\delta\hat{\Sigma}_{g,\overline{N};f,\overline{M}}^{(\alpha;\kappa)b\neq a}(x_p)}\Big)\frac{\begin{pmatrix}1&\\&\mathbf{+1}\end{pmatrix}^{aa}}{\hat{\mathfrak{v}}_{\hat{F};\overline{M}}^{(\alpha;\kappa)}(x_p)}\Bigg]\bigg\}\times \\
&\times\;\exp\bigg\{\frac{\imath}{4}\int_C d^4x_p\sum_{(\alpha)=1,..,8}^{(\kappa)=0,..,3}\mathop{\mathfrak{Tr}}_{N_f,\hat{\gamma}_{mn}^{(\mu)},N_c}^{a(=1,2)}\bigg[\Big(\delta\hat{\Sigma}_{M;N}^{(\alpha;\kappa)\boldsymbol{a\neq b}}(x_p)\Big)\begin{pmatrix}1&\\&-1\end{pmatrix}^{bb}\Big(\hat{\mathfrak{R}}_{N;M}^{(\alpha;\kappa)\boldsymbol{b\neq a}}(x_p)\Big)\begin{pmatrix}1&\\&-1\end{pmatrix}^{aa}\bigg]\bigg\}\,.
\end{aligned}
$$

The diagonalized, two-body potential part $[-\imath\hat{\mathfrak{e}}_p^{(\hat{F})}+\hat{\mathfrak{b}}_{(\alpha;\kappa)}^{(\hat{F})}(x_p)]$ and the eigenvalues $\hat{\mathfrak{v}}_{\hat{F};\overline{M}}^{(\alpha;\kappa)}(x_p)$, which replace the anomalous indexed potentials $\hat{\boldsymbol{\mathcal{V}}}_{(\alpha);f',\overline{M}';f,\overline{M}}^{(\hat{F})a;(\kappa)}(x_p)$ (3.17,3.18), have the effect of a kind of 'variance' in the Gaussian integrations of the BCS related quark matter self-energy terms so that the polynomials $\mathfrak{P}_1(\,\hat{\mathfrak{b}}_{(\alpha;\kappa)}^{(\hat{F})}(x_p)\,)$, $\mathfrak{Q}_1(\,\hat{\mathfrak{v}}_{\hat{F};\overline{M}}^{(\alpha;\kappa)}(x_p)\,)$ have to be incorporated in the Gaussian integrations of the HST's for a proper normalization.

The other half of the HST's for the quartic interaction[9] with the original dyadic product of anomalous doubled Fermi fields is obtained by the independent, 32, real, scalar self-energies $\sigma_D^{(\alpha;\kappa)}(x_p)$ (3.34,3.35) which are related to quark densities. Since the Gaussian integrals with $\sigma_D^{(\alpha;\kappa)}(x_p)$ are also weighted by the variance of the two-body related potential $[-\imath\,\hat{\mathfrak{e}}_p^{(\hat{F})}+\hat{\mathfrak{b}}_{(\alpha;\kappa)}^{(\hat{F})}(x_p)]$, we have to include an additional polynomial $\mathfrak{P}_2(\,\hat{\mathfrak{b}}_{(\alpha;\kappa)}^{(\hat{F})}(x_p)\,)$ for the normalization of the 32 HST's

$$
\begin{aligned}
&\exp\bigg\{\frac{\imath}{16}\int_C d^4x_p\,\Big(\Psi_{N'}^{\dagger,b}(x_p)\,\hat{\boldsymbol{\Gamma}}_{\gamma;\nu;N';N}^{bb}\,\hat{S}^{bb}\,\Psi_N^b(x_p)\Big)\Big[-\imath\,\hat{\mathfrak{e}}_p^{(\hat{F})}+C_{\alpha\beta'\gamma'}\,\hat{\mathfrak{G}}_\alpha^{(\hat{F})\mu'\nu'}(x_p)\Big]_{\gamma\beta}^{\mathbf{-1};\nu\mu}\times \qquad (3.37)\\
&\qquad\Big(\Psi_{M'}^{\dagger,a}(x_p)\,\hat{\boldsymbol{\Gamma}}_{\beta;\mu;M';M}^{aa}\,\kappa^{aa}\,\Psi_M^a(x_p)\Big)\bigg\}= \\
&= \int d[\sigma_D^{(\alpha;\kappa)}(x_p)]\;\mathfrak{P}_2\Big(\hat{\mathfrak{b}}_{(\alpha;\kappa)}^{(\hat{F})}(x_p)\Big)\;\exp\bigg\{\frac{\imath}{4}\int_C d^4x_p\sum_{(\alpha)=1,..,8}^{(\kappa)=0,..,3}\Big[\imath\,\hat{\mathfrak{e}}_p^{(\hat{F})}+\hat{\mathfrak{b}}_{(\alpha;\kappa)}^{(\hat{F})}(x_p)\Big]\Big(\sigma_D^{(\alpha;\kappa)}(x_p)\Big)^2\bigg\}\times \\
&\times\;\exp\bigg\{\frac{\imath}{4}\int_C d^4x_p\sum_{(\alpha)=1,..,8}^{(\kappa)=0,..,3}\mathop{\mathfrak{Tr}}_{N_f,\hat{\gamma}_{mn}^{(\mu)},N_c}^{a(=1,2)}\bigg[\begin{pmatrix}\hat{\mathfrak{R}}_{g,\overline{N};f,\overline{M}}^{(\alpha;\kappa)11}(x_p)&0\\0&\hat{\mathfrak{R}}_{g,\overline{N};f,\overline{M}}^{(\alpha;\kappa)22}(x_p)\end{pmatrix}\begin{pmatrix}1&\\&-1\end{pmatrix}\times \\
&\times\;\begin{pmatrix}\sigma_D^{(\alpha;\kappa)}(x_p)\,\hat{\mathfrak{v}}_{\hat{F};\overline{M}}^{(\alpha;\kappa)}(x_p)\,\delta_{\overline{M};\overline{N}}\,\delta_{f,g}&0\\0&-\sigma_D^{(\alpha;\kappa)}(x_p)\,\hat{\mathfrak{v}}_{\hat{F};\overline{M}}^{(\alpha;\kappa)}(x_p)\,\delta_{\overline{M};\overline{N}}\,\delta_{f,g}\end{pmatrix}\begin{pmatrix}1&\\&-1\end{pmatrix}\bigg]\bigg\}\,.
\end{aligned}
$$

Moreover, one has to introduce 'hinge' fields which allow for the coset decomposition $\mathrm{SO}(N_0\,,N_0)/\mathrm{U}(N_0)\otimes\mathrm{U}(N_0)$, $(N_0=N_f\times4_{\hat{\gamma}}\times N_c)$ as the subgroup part in the spontaneous symmetry breaking to BCS related pair condensates of quark fields. (QCD-type case with up-, down-isospins yields $\mathrm{SO}(24\,,24)/\mathrm{U}(24)\otimes\mathrm{U}(24)$ and with the inclusion of strangeness $\mathrm{SO}(36\,,36)/\mathrm{U}(36)\otimes\mathrm{U}(36)$.) In compliance with Ref. [11], we therefore take into account additional Gaussian integrations of self-energy densities $\delta\hat{\Sigma}_{M;N}^{(\alpha;\kappa)11}(x_p)$, $\delta\hat{\Sigma}_{M;N}^{(\alpha;\kappa)22}(x_p)$ (3.32,3.33) which are normalized to unity by the polynomials $\mathfrak{P}_3(\,\hat{\mathfrak{b}}_{(\alpha;\kappa)}^{(\hat{F})}(x_p)\,)$, $\mathfrak{Q}_3(\,\hat{\mathfrak{v}}_{\hat{F};\overline{M}}^{(\alpha;\kappa)}(x_p)\,)$. This normalization to unity is caused by the additional minus sign $-\delta\hat{\Sigma}_{M;N}^{(\alpha;\kappa)22}(x_p)$ in the '22' block density part which can be transformed to a non-vanishing '+' part $+\delta\hat{\Sigma}_{M;N}^{(\alpha;\kappa)22}(x_p)$, but then induces the required anti-hermitian BCS related parts in the off-diagonal '12', '21' blocks for the HST of half of the quartic interaction

[9]Note again the pre-factor $\imath/16$ instead of $\imath/8$ as in (3.27) for the 'half' of the quartic interaction of Fermi fields !

(3.36)

$$1 \equiv \int d[\delta\hat{\Sigma}_{M;N}^{(\alpha;\kappa)11}(x_p);\delta\hat{\Sigma}_{M;N}^{(\alpha;\kappa)22}(x_p)]\;\;\mathfrak{P}_3\Big(\hat{\mathfrak{b}}_{(\alpha;\kappa)}^{(\hat{F})}(x_p)\Big)\;\mathfrak{Q}_3\Big(\hat{\mathfrak{v}}_{\hat{F};\overline{M}}^{(\alpha;\kappa)}(x_p)\Big)\;\exp\Bigg\{\frac{\imath}{8}\int_C d^4x_p\sum_{(\alpha)=1,..,8}^{(\kappa)=0,..,3}\times \tag{3.38}$$
$$\times\;\;\Big[-\imath\,\hat{\mathfrak{e}}_p^{(\hat{F})}+\hat{\mathfrak{b}}_{(\alpha;\kappa)}^{(\hat{F})}(x_p)\Big]\;\underset{N_f,\hat{\gamma}_{mn}^{(\mu)},N_c}{\overset{a(=1,2)}{\mathfrak{Tr}}}\Bigg[\Big(\delta\hat{\Sigma}_{f,\overline{M};g,\overline{N}}^{(\alpha;\kappa)\boldsymbol{aa}}(x_p)\Big)\frac{1}{\hat{\mathfrak{v}}_{\hat{F};\overline{N}}^{(\alpha;\kappa)}(x_p)}\Big(\delta\hat{\Sigma}_{g,\overline{N};f,\overline{M}}^{(\alpha;\kappa)\boldsymbol{aa}}(x_p)\Big)\frac{1}{\hat{\mathfrak{v}}_{\hat{F};\overline{M}}^{(\alpha;\kappa)}(x_p)}\Bigg]\Bigg\}\times$$
$$\times\;\;\exp\Bigg\{\frac{\imath}{4}\int_C d^4x_p\sum_{(\alpha)=1,..,8}^{(\kappa)=0,..,3}\;\underset{N_f,\hat{\gamma}_{mn}^{(\mu)},N_c}{\overset{a(=1,2)}{\mathfrak{Tr}}}\Bigg[\begin{pmatrix}\hat{\mathfrak{R}}_{N;M}^{(\alpha;\kappa)11}(x_p) & 0\\ 0 & \hat{\mathfrak{R}}_{N;M}^{(\alpha;\kappa)22}(x_p)\end{pmatrix}\begin{pmatrix}1 & \\ & -1\end{pmatrix}\times$$
$$\times\;\;\begin{pmatrix}\delta\hat{\Sigma}_{M;N}^{(\alpha;\kappa)11}(x_p) & 0\\ 0 & \boldsymbol{-\delta\hat{\Sigma}_{M;N}^{(\alpha;\kappa)22}(x_p)}\end{pmatrix}\begin{pmatrix}1 & \\ & -1\end{pmatrix}\Bigg]\Bigg\}.$$

The combination of all subsequent parts eventually yields the entire HST (3.39) to self-energies of the quark matter fields

$$\exp\Bigg\{\frac{\imath}{8}\int_C d^4x_p\;\Big(\Psi_{N'}^{\dagger,b}(x_p)\;\hat{\boldsymbol{\Gamma}}_{\gamma;\nu;N';N}^{bb}\;\hat{S}^{bb}\;\Psi_N^b(x_p)\Big)\;\Big[-\imath\,\hat{\mathfrak{e}}_p^{(\hat{F})}+C_{\alpha\beta'\gamma'}\;\hat{\mathfrak{S}}_\alpha^{(\hat{F})\mu'\nu'}(x_p)\Big]_{\gamma\beta}^{\boldsymbol{-1};\nu\mu}\times \tag{3.39}$$
$$\Big(\Psi_{M'}^{\dagger,a}(x_p)\;\hat{\boldsymbol{\Gamma}}_{\beta;\mu;M';M}^{aa}\;\kappa^{aa}\;\Psi_M^a(x_p)\Big)\Bigg\}=$$
$$=\;\int d[\sigma_D^{(\alpha;\kappa)}(x_p)]\;\;\mathfrak{P}_2\Big(\hat{\mathfrak{b}}_{(\alpha;\kappa)}^{(\hat{F})}(x_p)\Big)\;\exp\Bigg\{\frac{\imath}{4}\int_C d^4x_p\sum_{(\alpha)=1,..,8}^{(\kappa)=0,..,3}\Big[\imath\,\hat{\mathfrak{e}}_p^{(\hat{F})}+\hat{\mathfrak{b}}_{(\alpha;\kappa)}^{(\hat{F})}(x_p)\Big]\;\Big(\sigma_D^{(\alpha;\kappa)}(x_p)\Big)^2\Bigg\}\times$$
$$\times\;\;\int d[\delta\widetilde{\Sigma}_{M;N}^{(\alpha;\kappa)ab}(x_p)]\;\;\mathfrak{P}_1\Big(\hat{\mathfrak{b}}_{(\alpha;\kappa)}^{(\hat{F})}(x_p)\Big)\;\mathfrak{P}_3\Big(\hat{\mathfrak{b}}_{(\alpha;\kappa)}^{(\hat{F})}(x_p)\Big)\;\;\mathfrak{Q}_1\Big(\hat{\mathfrak{v}}_{\hat{F};\overline{M}}^{(\alpha;\kappa)}(x_p)\Big)\;\mathfrak{Q}_3\Big(\hat{\mathfrak{v}}_{\hat{F};\overline{M}}^{(\alpha;\kappa)}(x_p)\Big)\;\times$$
$$\times\;\;\exp\Bigg\{\frac{\imath}{8}\int_C d^4x_p\sum_{(\alpha)=1,..,8}^{(\kappa)=0,..,3}\Big[-\imath\,\hat{\mathfrak{e}}_p^{(\hat{F})}+\hat{\mathfrak{b}}_{(\alpha;\kappa)}^{(\hat{F})}(x_p)\Big]\;\times$$
$$\times\underset{N_f,\hat{\gamma}_{mn}^{(\mu)},N_c}{\overset{a(=1,2)}{\mathfrak{Tr}}}\Bigg[\delta\widetilde{\Sigma}_{f,\overline{M};g,\overline{N}}^{(\alpha;\kappa)ab}(x_p)\;\frac{1}{\hat{\mathfrak{v}}_{\hat{F};\overline{N}}^{(\alpha;\kappa)}(x_p)}\;\delta\widetilde{\Sigma}_{g,\overline{N};f,\overline{M}}^{(\alpha;\kappa)ba}(x_p)\;\frac{1}{\hat{\mathfrak{v}}_{\hat{F};\overline{M}}^{(\alpha;\kappa)}(x_p)}\Bigg]\Bigg\}\exp\Bigg\{\frac{\imath}{4}\int_C d^4x_p\sum_{(\alpha)=1,..,8}^{(\kappa)=0,..,3}\times$$
$$\times\underset{N_f,\hat{\gamma}_{mn}^{(\mu)},N_c}{\overset{a(=1,2)}{\mathfrak{Tr}}}\Bigg[\begin{pmatrix}\hat{\mathfrak{R}}_{N;M}^{(\alpha;\kappa)11}(x_p) & \hat{\mathfrak{R}}_{N;M}^{(\alpha;\kappa)12}(x_p)\\ \hat{\mathfrak{R}}_{N;M}^{(\alpha;\kappa)21}(x_p) & \hat{\mathfrak{R}}_{N;M}^{(\alpha;\kappa)22}(x_p)\end{pmatrix}\begin{pmatrix}1 & \\ & -1\end{pmatrix}\begin{pmatrix}\hat{\Sigma}_{M;N}^{(\alpha;\kappa)11}(x_p) & \delta\hat{\Sigma}_{M;N}^{(\alpha;\kappa)12}(x_p)\\ \delta\hat{\Sigma}_{M;N}^{(\alpha;\kappa)21}(x_p) & -\hat{\Sigma}_{M;N}^{(\alpha;\kappa)22}(x_p)\end{pmatrix}\begin{pmatrix}1 & \\ & -1\end{pmatrix}\Bigg]\Bigg\}.$$

The entire HST (3.39) for the quartic interaction of fermionic fields can be inserted into the generating function (3.8,3.9) $Z[\hat{\mathcal{J}},J_\psi,\hat{J}_{\psi\psi},\hat{\mathfrak{j}}^{(\hat{F})}]$ so that we obtain relation (3.40) which depends on the 32 independent self-energies $\delta\widetilde{\Sigma}_{M;N}^{(\alpha;\kappa)ab}(x_p)$ of quark matter fields (also with anomalous pairing), on the 32 independent, real, diagonal, scalar self-energy density fields $\sigma_D^{(\alpha;\kappa)}(x_p)$ and remaining Gaussian integrations of anti-commuting variables $d[\psi_M^\dagger(x_p),\,\psi_M(x_p)]$

$$Z[\hat{\mathcal{J}},J_\psi,\hat{J}_{\psi\psi},\hat{\mathfrak{j}}^{(\hat{F})}]=\Bigg\langle\boldsymbol{Z\Big[\hat{\mathfrak{S}}^{(\hat{F})},s_\alpha,\hat{\mathfrak{B}}_{\hat{F}},\hat{\mathfrak{b}}^{(\hat{F})},\hat{\mathfrak{U}}_{\hat{F}},\hat{\mathfrak{v}}_{\hat{F}};\hat{\mathfrak{j}}^{(\hat{F})};\text{Eq. (3.42)}\Big]}\times \tag{3.40}$$
$$\times\;\;\int d[\sigma_D^{(\alpha;\kappa)}(x_p)]\;\exp\Bigg\{\frac{\imath}{4}\int_C d^4x_p\sum_{(\alpha)=1,..,8}^{(\kappa)=0,..,3}\Big[\imath\,\hat{\mathfrak{e}}_p^{(\hat{F})}+\hat{\mathfrak{b}}_{(\alpha;\kappa)}^{(\hat{F})}(x_p)\Big]\;\Big(\sigma_D^{(\alpha;\kappa)}(x_p)\Big)^2\Bigg\}\times$$
$$\times\;\;\int d[\delta\widetilde{\Sigma}_{M;N}^{(\alpha;\kappa)ab}(x_p)]\;\exp\Bigg\{\frac{\imath}{8}\int_C d^4x_p\sum_{(\alpha)=1,..,8}^{(\kappa)=0,..,3}\Big[-\imath\,\hat{\mathfrak{e}}_p^{(\hat{F})}+\hat{\mathfrak{b}}_{(\alpha;\kappa)}^{(\hat{F})}(x_p)\Big]\;\times$$

$$\times \underset{N_f,\hat{\gamma}_{mn}^{(\mu)},N_c}{\overset{a(=1,2)}{\mathfrak{Tr}}} \bigg[\delta\widetilde{\Sigma}_{f,\overline{M};g,\overline{N}}^{(\alpha;\kappa)ab}(x_p)\;\frac{1}{\hat{\mathfrak{v}}_{\hat{F};\overline{N}}^{(\alpha;\kappa)}(x_p)}\;\delta\widetilde{\Sigma}_{g,\overline{N};f,\overline{M}}^{(\alpha;\kappa)ba}(x_p)\;\frac{1}{\hat{\mathfrak{v}}_{\hat{F};\overline{M}}^{(\alpha;\kappa)}(x_p)}\bigg]\bigg\}\times$$
$$\times \int d[\psi_M^\dagger(x_p),\,\psi_M(x_p)]\;\exp\bigg\{-\frac{\imath}{2}\int_C d^4x_p\;d^4y_q\;\Psi_N^{\dagger,b}(y_q)\;\widetilde{M}_{N;M}^{ba}(y_q,x_p)\;\Psi_M^a(x_p)\bigg\}\times$$
$$\times \exp\bigg\{-\frac{\imath}{2}\int_C d^4x_p\;\Big(J_{\psi;M}^{\dagger,a}(x_p)\;\hat{S}^{ab}\;\Psi_M^b(x_p)+\Psi_M^{\dagger,a}(x_p)\;\hat{S}^{ab}\;J_{\psi;M}^b(x_p)\Big)\bigg\}\bigg\rangle\,.$$

Relation (3.40) is simplified by introducing the matrix $\widetilde{M}_{N;M}^{ba}(y_q,x_p)$ (3.41) and the background generating function (3.42) of the gauge field propagation. Apart from the sources $\hat{\jmath}_{N;M}^{ba}(y_q,x_p)$, $\hat{J}_{\psi\psi;N;M}^{b\neq a}(x_p)$ and the one-particle Hamiltonian, the matrix $\widetilde{M}_{N;M}^{ba}(y_q,x_p)$ (3.41) contains the gauge field strength self-energy $\hat{\mathfrak{G}}_{\alpha;\mu\nu}^{(\hat{F})}(x_p)$, the structure constants $C_{\alpha\beta\gamma}$ of $\mathrm{SU}_c(N_c=3)$ and the quark self-energy density fields $\sigma_D^{(\alpha;\kappa)}(x_p)$ coupled to diagonalizing matrices $\hat{\mathfrak{U}}_{\hat{F};\overline{M}';\overline{M}}^{(\alpha;\kappa),aa}(x_p)$, $\hat{\mathfrak{U}}_{\hat{F};\overline{N};\overline{M}'}^{(\alpha;\kappa),bb,\dagger}(x_p)$ with eigenvalues $\hat{\mathfrak{v}}_{\hat{F};\overline{M}'}^{(\alpha;\kappa)}(x_p)$ of the anomalous indexed potentials $\hat{\boldsymbol{\mathcal{V}}}_{(\alpha);f',\overline{M}';f,\overline{M}}^{(\hat{F})a;(\kappa)}(x_p)$ (3.17,3.18). We assume that these gauge field related terms create a confining potential for the anomalous doubled self-energies $\delta\widetilde{\Sigma}_{M;N}^{(\alpha;\kappa)ab}(x_p)$ of quark matter fields where the confinement potential is also partially determined by the source fields $J_{\psi;M}^a(x_p)$, $\hat{J}_{\psi\psi;N;M}^{b\neq a}(x_p)$. After using Eqs. (3.21-3.25), we can also link the quark self-energy density fields $\sigma_D^{(\alpha;\kappa)}(x_p)$ and the term with the self-energy of the gauge field strength tensor together into a single potential part so that the relevant terms for a confinement become more obvious

$$\begin{aligned}
&\widetilde{M}_{N;M}^{ba}(y_q,x_p)=\hat{\jmath}_{N;M}^{ba}(y_q,x_p)+\delta^{(4)}(y_q-x_p)\;\delta_{pq}\;\eta_q\times \qquad (3.41)\\
&\times\;\Bigg[\Bigg(\begin{array}{cc}\big[\hat{\beta}(\hat{\gamma}^\mu\hat{\partial}_{p,\mu}-\imath\,\hat{\varepsilon}_p+\hat{m})\big]_{N;M} & \hat{\jmath}_{\psi\psi;N;M}(x_p)\\ \hat{\jmath}_{\psi\psi;N;M}^\dagger(x_p) & -\big[\hat{\beta}(\hat{\gamma}^\mu\hat{\partial}_{p,\mu}-\imath\,\hat{\varepsilon}_p+\hat{m})\big]_{N;M}^T\end{array}\Bigg)_{N;M}^{ba}+\\
&+\;\Big[\Big(\hat{\partial}_p^\lambda\hat{\mathfrak{G}}_{\gamma;\nu\lambda}^{(\hat{F})}(x_p)\Big)-s_\gamma(x_p)\;n_\nu\Big]\;\Big[-\imath\,\hat{\mathfrak{e}}_p^{(\hat{F})}+C_{\alpha\beta'\gamma'}\;\hat{\mathfrak{G}}_\alpha^{(\hat{F})\mu'\nu'}(x_p)\Big]_{\gamma\beta}^{\boldsymbol{-1};\nu\mu}\times\\
&\times\;\Bigg(\begin{array}{cc}\big[\hat{\beta}(\imath\;\hat{\gamma}_\mu\;\hat{t}_\beta)\big]_{\overline{N};\overline{M}} & 0\\ 0 & -\big[\hat{\beta}(\imath\;\hat{\gamma}_\mu\;\hat{t}_\beta)\big]_{\overline{N};\overline{M}}^T\end{array}\Bigg)_{g,\overline{N};f,\overline{M}}^{ba}\delta_{g,f}+\\
&+\;\sum_{(\alpha)=1,..,8}^{(\kappa)=0,..,3}\Bigg(\frac{\hat{S}^{bb}}{2}\;\sigma_D^{(\alpha;\kappa)}(x_p)\;\delta_{ab}\;\hat{\mathfrak{U}}_{\hat{F};\overline{N};\overline{M}'}^{(\alpha;\kappa),bb,\dagger}(x_p)\;\;\hat{\mathfrak{v}}_{\hat{F};\overline{M}'}^{(\alpha;\kappa)}(x_p)\;\;\hat{\mathfrak{U}}_{\hat{F};\overline{M}';\overline{M}}^{(\alpha;\kappa),aa}(x_p)\;\;\delta_{g,f}+\\
&+\;\frac{1}{2}\hat{S}^{bb}\;\;\hat{\mathfrak{U}}_{\hat{F};\overline{N};\overline{N}'}^{(\alpha;\kappa),bb,\dagger}(x_p)\;\Bigg(\begin{array}{cc}\delta\hat{\Sigma}_{g,\overline{N}';f,\overline{M}'}^{(\alpha;\kappa)11}(x_p) & \delta\hat{\Sigma}_{g,\overline{N}';f,\overline{M}'}^{(\alpha;\kappa)12}(x_p)\\ \delta\hat{\Sigma}_{g,\overline{N}';f,\overline{M}'}^{(\alpha;\kappa)21}(x_p) & -\delta\hat{\Sigma}_{g,\overline{N}';f,\overline{M}'}^{(\alpha;\kappa)22}(x_p)\end{array}\Bigg)_{N';M'}^{ba}\hat{\mathfrak{U}}_{\hat{F};\overline{M}';\overline{M}}^{(\alpha;\kappa),aa}(x_p)\;\;\hat{S}^{aa}\Bigg)\Bigg]_{g,\overline{N};f,\overline{M}}^{ba}=\\
&=\;\hat{\jmath}_{N;M}^{ba}(y_q,x_p)+\delta^{(4)}(y_q-x_p)\;\delta_{pq}\;\eta_q\Bigg[\Bigg(\begin{array}{cc}\big[\hat{\beta}(\hat{\gamma}^\mu\hat{\partial}_{p,\mu}-\imath\,\hat{\varepsilon}_p+\hat{m})\big]_{N;M} & \hat{\jmath}_{\psi\psi;N;M}(x_p)\\ \hat{\jmath}_{\psi\psi;N;M}^\dagger(x_p) & -\big[\hat{\beta}(\hat{\gamma}^\mu\hat{\partial}_{p,\mu}-\imath\,\hat{\varepsilon}_p+\hat{m})\big]_{N;M}^T\end{array}\Bigg)_{N;M}^{ba}+\\
&+\Bigg(\Big[\Big(\hat{\partial}_p^\lambda\hat{\mathfrak{G}}_{\gamma;\nu\lambda}^{(\hat{F})}(x_p)\Big)-s_\gamma(x_p)\;n_\nu\Big]\Big[-\imath\,\hat{\mathfrak{e}}_p^{(\hat{F})}+C_{\alpha\beta'\gamma'}\;\hat{\mathfrak{G}}_\alpha^{(\hat{F})\mu'\nu'}(x_p)\Big]_{\gamma\beta}^{\boldsymbol{-1};\nu\mu}+\frac{1}{2}\sum_{(\alpha)=1,..,8}^{(\kappa)=0,..,3}\hat{\mathfrak{B}}_{\hat{F};\beta(\alpha)}^{\mu(\kappa)}(x_p)\;\sigma_D^{(\alpha;\kappa)}(x_p)\Bigg)_\beta^\mu\times\\
&\times\;\Bigg(\begin{array}{cc}\big[\hat{\beta}(\imath\;\hat{\gamma}_\mu\;\hat{t}_\beta)\big]_{\overline{N};\overline{M}} & 0\\ 0 & -\big[\hat{\beta}(\imath\;\hat{\gamma}_\mu\;\hat{t}_\beta)\big]_{\overline{N};\overline{M}}^T\end{array}\Bigg)_{g,\overline{N};f,\overline{M}}^{ba}\delta_{g,f}+\\
&+\frac{1}{2}\hat{S}^{bb}\sum_{(\alpha)=1,..,8}^{(\kappa)=0,..,3}\hat{\mathfrak{U}}_{\hat{F};\overline{N};\overline{N}'}^{(\alpha;\kappa),bb,\dagger}(x_p)\;\Bigg(\begin{array}{cc}\delta\hat{\Sigma}_{g,\overline{N}';f,\overline{M}'}^{(\alpha;\kappa)11}(x_p) & \delta\hat{\Sigma}_{g,\overline{N}';f,\overline{M}'}^{(\alpha;\kappa)12}(x_p)\\ \delta\hat{\Sigma}_{g,\overline{N}';f,\overline{M}'}^{(\alpha;\kappa)21}(x_p) & -\delta\hat{\Sigma}_{g,\overline{N}';f,\overline{M}'}^{(\alpha;\kappa)22}(x_p)\end{array}\Bigg)^{ba}\hat{\mathfrak{U}}_{\hat{F};\overline{M}';\overline{M}}^{(\alpha;\kappa),aa}(x_p)\;\hat{S}^{aa}\Bigg)\Bigg]_{g,\overline{N};f,\overline{M}}^{ba}.
\end{aligned}$$

The background path integral for (3.40) is listed in (3.42) and also incorporates the delta-functions (3.28) of the diagonalization (3.10,3.22) and the real, auxiliary field $s_\alpha(x_p)$ for the axial gauge fixing. Furthermore, we have to include the

polynomials $\mathfrak{P}_1(\,\hat{\mathfrak{b}}_{\alpha;\kappa}^{(\hat{F})}(x_p)\,)$, $\mathfrak{P}_2(\,\hat{\mathfrak{b}}_{\alpha;\kappa}^{(\hat{F})}(x_p)\,)$, $\mathfrak{P}_3(\,\hat{\mathfrak{b}}_{\alpha;\kappa}^{(\hat{F})}(x_p)\,)$, $\mathfrak{Q}_1(\,\hat{\mathfrak{v}}_{\hat{F};\overline{M}}^{(\alpha;\kappa)}(x_p)\,)$, $\mathfrak{Q}_3(\,\hat{\mathfrak{v}}_{\hat{F};\overline{M}}^{(\alpha;\kappa)}(x_p)\,)$ which have been introduced for proper normalization of Gaussian integrals in the HST's

$$\begin{aligned}
&\bigg\langle \boldsymbol{Z}\Big[\hat{\boldsymbol{\mathfrak{S}}}^{(\hat{F})},\boldsymbol{s_\alpha},\boldsymbol{\mathfrak{B}_{\hat{F}}},\hat{\boldsymbol{\mathfrak{b}}}^{(\hat{F})},\hat{\boldsymbol{\mathfrak{U}}}_{\hat{F}},\hat{\boldsymbol{\mathfrak{v}}}_{\hat{F}};\hat{\boldsymbol{\mathfrak{j}}}^{(\hat{F})};\textbf{Eq. (3.42)}\Big]\Big(\textbf{fields}\Big)\bigg\rangle = \qquad (3.42)\\
&= \bigg\langle\int d[\hat{\mathfrak{S}}_\alpha^{(\hat{F})\mu\nu}(x_p)]\;\; d[\hat{\mathfrak{B}}_{\hat{F};\beta(\alpha)}^{\mu(\kappa)}(x_p);\hat{\mathfrak{b}}_{(\alpha;\kappa)}^{(\hat{F})}(x_p)]\;\; d[\hat{\mathfrak{U}}_{\hat{F};\overline{N};\overline{M}}^{(\alpha;\kappa)}(x_p);\hat{\mathfrak{v}}_{\hat{F};\overline{N}}^{(\alpha;\kappa)}(x_p)]\;\times\\
&\times\;\;\Big\{\prod_{\{x_p\}}\delta\Big(C_{\alpha\beta\gamma}\;\hat{\mathfrak{S}}_\alpha^{(\hat{F})\mu\nu}(x_p)-\sum_{(\alpha)=1,..,8}^{(\kappa)=0,..,3}\hat{\mathfrak{B}}_{\hat{F};\beta(\alpha)}^{\mu(\kappa)}(x_p)\;\hat{\mathfrak{b}}_{(\alpha;\kappa)}^{(\hat{F})}(x_p)\;\hat{\mathfrak{B}}_{\hat{F};(\alpha)\gamma}^{T,(\kappa)\nu}(x_p)\Big)\Big\}\times\\
&\times\;\;\Big\{\prod_{\{x_p\};(\alpha)=1,..,8}^{\{(\kappa)=0,..,3\}}\delta\Big(\hat{\mathfrak{U}}_{\hat{F};\overline{M}';\overline{N}}^{(\alpha;\kappa),11,\dagger}(x_p)\;\hat{\mathfrak{v}}_{\hat{F};\overline{N}}^{(\alpha;\kappa)}(x_p)\;\hat{\mathfrak{U}}_{\hat{F};\overline{N};\overline{M}}^{(\alpha;\kappa),11}(x_p)-\big[\hat{\beta}\,(\imath\,\hat{\gamma}_\mu\;\hat{t}_\beta)\big]_{\overline{M}';\overline{M}}\;\hat{\mathfrak{B}}_{\hat{F};\beta(\alpha)}^{\mu(\kappa)}(x_p)\Big)\Big\}\times\\
&\times\;\;\mathfrak{P}_1\Big(\hat{\mathfrak{b}}_{(\alpha;\kappa)}^{(\hat{F})}(x_p)\Big)\;\mathfrak{P}_2\Big(\hat{\mathfrak{b}}_{(\alpha;\kappa)}^{(\hat{F})}(x_p)\Big)\;\mathfrak{P}_3\Big(\hat{\mathfrak{b}}_{(\alpha;\kappa)}^{(\hat{F})}(x_p)\Big)\;\;\mathfrak{Q}_1\Big(\hat{\mathfrak{v}}_{\hat{F};\overline{M}}^{(\alpha;\kappa)}(x_p)\Big)\;\mathfrak{Q}_3\Big(\hat{\mathfrak{v}}_{\hat{F};\overline{M}}^{(\alpha;\kappa)}(x_p)\Big)\;\int d[s_\alpha(x_p)]\;\times\\
&\times\;\;\exp\Big\{\imath\int_C d^4\!x_p\Big(\frac{1}{4}\hat{\mathfrak{S}}_\alpha^{(\hat{F})\mu\nu}(x_p)\;\hat{\mathfrak{S}}_{\alpha;\mu\nu}^{(\hat{F})}(x_p)-\hat{\mathfrak{j}}_{\alpha;\mu\nu}^{(\hat{F})}(x_p)\;\hat{\mathfrak{S}}_\alpha^{(\hat{F})\mu\nu}(x_p)+\hat{\mathfrak{j}}_{\alpha;\mu\nu}^{(\hat{F})}(x_p)\;\hat{\mathfrak{j}}_\alpha^{(\hat{F})\mu\nu}(x_p)\Big)\Big\}\times\\
&\times\;\;\Big\{\det\Big[\Big(-\imath\,\hat{\mathfrak{e}}_p^{(\hat{F})}+C_{\alpha\beta\gamma}\;\hat{\mathfrak{S}}_\alpha^{(\hat{F})\mu\nu}(x_p)\Big)_{\beta\gamma}^{\mu\nu}\Big]\Big\}^{\boldsymbol{-1/2}}\times\exp\Big\{\frac{\imath}{2}\int_C d^4\!x_p\;\Big[\big(\hat{\partial}_p^\lambda\hat{\mathfrak{S}}_{\gamma;\nu\lambda}^{(\hat{F})}(x_p)\big)-s_\gamma(x_p)\,n_\nu\Big]\;\times\\
&\times\;\;\Big[-\imath\,\hat{\mathfrak{e}}_p^{(\hat{F})}+C_{\alpha\beta'\gamma'}\;\hat{\mathfrak{S}}_\alpha^{(\hat{F})\mu'\nu'}(x_p)\Big]_{\gamma\beta}^{\boldsymbol{-1};\nu\mu}\;\Big[\big(\hat{\partial}_p^\kappa\hat{\mathfrak{S}}_{\beta;\mu\kappa}^{(\hat{F})}(x_p)\big)-s_\beta(x_p)\,n_\mu\Big]\Big\}\;\times\Big(\textbf{fields}\Big)\bigg\rangle\,.
\end{aligned}$$

3.3 Coset decomposition for BCS pair condensate degrees of freedom

The HST's of sections 3.1, 3.2 have transformed the original path integral (2.25-2.27) to $Z[\hat{\mathscr{J}},J_\psi,\hat{J}_{\psi\psi},\hat{\mathfrak{j}}^{(\hat{F})}]$ (3.8,3.9) and finally to (3.40) with matrix $\widetilde{M}_{N;M}^{ba}(y_q,x_p)$ (3.41) and background generating functional (3.42) of the gauge field degrees of freedom. If we disregard the anomalous doubled one-particle Hamiltonian, the quark self-energy densities $\sigma_D^{(\alpha;\kappa)}(x_p)$ and the gauge field strength tensor $\hat{\mathfrak{S}}_{\alpha;\mu\nu}^{(\hat{F})}(x_p)$, the generating function $Z[\hat{\mathscr{J}},J_\psi,\hat{J}_{\psi\psi},\hat{\mathfrak{j}}^{(\hat{F})}]$ (3.40) mainly consists of the sum of $(N_c^2-1=8)\times(3+1)=32$ anomalous doubled self-energies $\delta\widetilde{\Sigma}_{N;M}^{(\alpha;\kappa)ba}(x_p)$ in $\widetilde{M}_{N;M}^{ba}(y_q,x_p)$ (3.41) for the fermionic matter fields; the latter are also dressed by the block diagonal unitary matrices $\hat{\mathfrak{U}}_{\hat{F};\overline{N};\overline{N}'}^{(\alpha;\kappa),bb,\dagger}(x_p)$, $\hat{\mathfrak{U}}_{\hat{F};\overline{M}';\overline{M}}^{(\alpha;\kappa),aa}(x_p)$ with colour degrees of freedom. Therefore, we can introduce a single, anomalous doubled self-energy $\delta\widetilde{\Sigma}_{M;N}^{ab}(x_p)$ (3.44) with anti-hermitian, BCS related terms in the off-diagonal blocks $a\neq b$ for the entire sum of 32, colour dressed self-energies $\delta\widetilde{\Sigma}_{M;N}^{(\alpha;\kappa)ab}(x_p)$ in $\widetilde{M}_{N;M}^{ba}(y_q,x_p)$ (3.41). This is accomplished by including the delta-function (3.43) with $\delta\widetilde{\Sigma}_{M;N}^{ab}(x_p)$ for the sum of the 32, anomalous doubled, colour dressed self-energies $\delta\widetilde{\Sigma}_{M;N}^{(\alpha;\kappa)ab}(x_p)$; one has also to reckon the similar symmetries (3.45) for the density- and anomalous-related blocks of $\delta\widetilde{\Sigma}_{M;N}^{ab}(x_p)$ as those for the independent, 32 self-energies $\delta\widetilde{\Sigma}_{M;N}^{(\alpha;\kappa)ab}(x_p)$ (3.29-3.35)

$$\begin{aligned}
&\int d[\delta\widetilde{\Sigma}_{M;N}^{ab}(x_p)]\;\delta\Big(\delta\widetilde{\Sigma}_{f,\overline{M};g,\overline{N}}^{ab}(x_p)-\sum_{(\alpha)=1,..,8}^{(\kappa)=0,..,3}\hat{\mathfrak{U}}_{\hat{F};\overline{M};\overline{M}'}^{(\alpha;\kappa),aa,\dagger}(x_p)\;\delta\widetilde{\Sigma}_{f,\overline{M}';g,\overline{N}'}^{(\alpha;\kappa)ab}(x_p)\;\hat{\mathfrak{U}}_{\hat{F};\overline{N}';\overline{N}}^{(\alpha;\kappa),bb}(x_p)\Big)= \qquad (3.43)\\
&\qquad=\int d[\delta\widetilde{\Sigma}_{M;N}^{ab}(x_p)]\;\;\boldsymbol{\delta}\Bigg[\begin{pmatrix}\delta\hat{\Sigma}_{M;N}^{11}(x_p) & \imath\,\delta\hat{\Sigma}_{M;N}^{12}(x_p)\\ \imath\,\delta\hat{\Sigma}_{M;N}^{21}(x_p) & \delta\hat{\Sigma}_{M;N}^{22}(x_p)\end{pmatrix}_{f,\overline{M};g,\overline{N}}^{ab}+\\
&\qquad-\sum_{(\alpha)=1,..,8}^{(\kappa)=0,..,3}\hat{\mathfrak{U}}_{\hat{F};\overline{M};\overline{M}'}^{(\alpha;\kappa),aa,\dagger}(x_p)\begin{pmatrix}\delta\hat{\Sigma}_{M';N'}^{(\alpha;\kappa)11}(x_p) & \imath\,\delta\hat{\Sigma}_{M';N'}^{(\alpha;\kappa)12}(x_p)\\ \imath\,\delta\hat{\Sigma}_{M';N'}^{(\alpha;\kappa)21}(x_p) & \delta\hat{\Sigma}_{M';N'}^{(\alpha;\kappa)22}(x_p)\end{pmatrix}_{f,\overline{M}';g,\overline{N}'}^{ab}\hat{\mathfrak{U}}_{\hat{F};\overline{N}';\overline{N}}^{(\alpha;\kappa),bb}(x_p)\Bigg]\,;
\end{aligned}$$

$$\delta\widetilde{\Sigma}_{M;N}^{ab}(x_p) = \begin{pmatrix} \delta\hat{\Sigma}_{M;N}^{11}(x_p) & \imath\,\delta\hat{\Sigma}_{M;N}^{12}(x_p) \\ \imath\,\delta\hat{\Sigma}_{M;N}^{21}(x_p) & \delta\hat{\Sigma}_{M;N}^{22}(x_p) \end{pmatrix}_{M;N}^{ab} ; \tag{3.44}$$

$$\begin{aligned} \Big(\delta\hat{\Sigma}_{M;N}^{11}(x_p)\Big)^{\dagger} &= \delta\hat{\Sigma}_{M;N}^{11}(x_p)\,; & \Big(\delta\hat{\Sigma}_{M;N}^{22}(x_p)\Big)^{\dagger} &= \delta\hat{\Sigma}_{M;N}^{22}(x_p)\,; \\ \delta\hat{\Sigma}_{M;N}^{22}(x_p) &= -\Big(\delta\hat{\Sigma}_{M;N}^{11}(x_p)\Big)^{T}\,; & \delta\hat{\Sigma}_{M;N}^{21}(x_p) &= \Big(\delta\hat{\Sigma}_{M;N}^{12}(x_p)\Big)^{\dagger}\,; \\ \delta\hat{\Sigma}_{M;N}^{12}(x_p) &= -\Big(\delta\hat{\Sigma}_{M;N}^{12}(x_p)\Big)^{T}\,; & \delta\hat{\Sigma}_{M;N}^{21}(x_p) &= -\Big(\delta\hat{\Sigma}_{M;N}^{21}(x_p)\Big)^{T}\,. \end{aligned} \tag{3.45}$$

The transformation (3.43-3.45) to a single, anomalous doubled self-energy for the fermionic fields considerably simplifies the matrix $\widetilde{M}_{N;M}^{ba}(y_q,x_p)$ (3.41) in the path integral (3.40,3.42). This is possible according to the analogous symmetries (3.45) of $\delta\widetilde{\Sigma}_{M;N}^{ab}(x_p)$ (3.44) as those of $\delta\widetilde{\Sigma}_{M;N}^{(\alpha;\kappa)ab}(x_p)$ (3.29-3.35). After inserting the delta-function (3.43) into (3.40-3.42), we integrate over the original, anti-commuting quark fields $d[\psi_M^{\dagger}(x_p),\,\psi_M(x_p)]$ in (3.40) and obtain the square root of the anomalous doubled Fermi determinant and the bilinear source term with $J_{\psi;N}^{\dagger,b}(y_q)$, $J_{\psi;M}^{a}(x_p)$ for the inverse of the simplified matrix $\widetilde{M}_{N;M}^{-\mathbf{1};ba}(y_q,x_p)$ (3.47)

$$\begin{aligned} & Z[\hat{\mathcal{J}},J_\psi,\hat{J}_{\psi\psi},\hat{\mathfrak{j}}^{(\hat{F})}] = \bigg\langle \boldsymbol{Z}\Big[\hat{\boldsymbol{\mathfrak{S}}}^{(\hat{F})},\boldsymbol{s_\alpha},\hat{\boldsymbol{\mathfrak{B}}}_{\hat{F}},\hat{\boldsymbol{\mathfrak{b}}}^{(\hat{F})},\hat{\boldsymbol{\mathfrak{U}}}_{\hat{F}},\hat{\boldsymbol{\mathfrak{v}}}_{\hat{F}};\hat{\boldsymbol{\mathfrak{j}}}^{(\hat{F})};\textbf{Eq. (3.42)}\Big] \times \\ & \times \int d[\sigma_D^{(\alpha;\kappa)}(x_p)]\;\exp\bigg\{\frac{\imath}{4}\int_C d^4x_p\sum_{(\alpha)=1,..,8}^{(\kappa)=0,..,3}\Big[\imath\,\hat{\mathfrak{e}}_p^{(\hat{F})}+\hat{\mathfrak{b}}_{(\alpha;\kappa)}^{(\hat{F})}(x_p)\Big]\Big(\sigma_D^{(\alpha;\kappa)}(x_p)\Big)^2\bigg\}\times \\ & \times \int d[\delta\widetilde{\Sigma}_{M;N}^{(\alpha;\kappa)ab}(x_p)]\;\exp\bigg\{\frac{\imath}{8}\int_C d^4x_p\sum_{(\alpha)=1,..,8}^{(\kappa)=0,..,3}\Big[-\imath\,\hat{\mathfrak{e}}_p^{(\hat{F})}+\hat{\mathfrak{b}}_{(\alpha;\kappa)}^{(\hat{F})}(x_p)\Big]\times \\ & \times \underset{N_f,\hat{\gamma}_{mn}^{(\mu)},N_c}{\overset{a(=1,2)}{\mathfrak{Tr}}}\bigg[\delta\widetilde{\Sigma}_{f,\overline{M};g,\overline{N}}^{(\alpha;\kappa)ab}(x_p)\,\frac{1}{\hat{\mathfrak{v}}_{\hat{F};\overline{N}}^{(\alpha;\kappa)}(x_p)}\;\delta\widetilde{\Sigma}_{g,\overline{N};f,\overline{M}}^{(\alpha;\kappa)ba}(x_p)\frac{1}{\hat{\mathfrak{v}}_{\hat{F};\overline{M}}^{(\alpha;\kappa)}(x_p)}\bigg]\bigg\}\times \\ & \times \int d[\delta\widetilde{\Sigma}_{M;N}^{ab}(x_p)]\;\delta\bigg(\delta\widetilde{\Sigma}_{f,\overline{M};g,\overline{N}}^{ab}(x_p)-\sum_{(\alpha)=1,..,8}^{(\kappa)=0,..,3}\hat{\mathfrak{U}}_{\hat{F};\overline{M};\overline{M}'}^{(\alpha;\kappa),aa,\dagger}(x_p)\;\delta\widetilde{\Sigma}_{f,\overline{M}';g,\overline{N}'}^{(\alpha;\kappa)ab}(x_p)\;\hat{\mathfrak{U}}_{\hat{F};\overline{N}';\overline{N}}^{(\alpha;\kappa),bb}(x_p)\bigg)\times \\ & \times \Big\{\mathrm{DET}\Big[\widetilde{M}_{N;M}^{ba}(y_q,x_p)\Big]\Big\}^{1/2}\exp\bigg\{\frac{\imath}{2}\int_C d^4x_p\,d^4y_q\;J_{\psi;N}^{\dagger,b}(y_q)\;\hat{S}^{bb}\;\widetilde{M}_{N;M}^{-\mathbf{1};ba}(y_q,x_p)\;\hat{S}^{aa}\;J_{\psi;M}^{a}(x_p)\bigg\}\bigg\rangle\,; \end{aligned} \tag{3.46}$$

$$\begin{aligned} & \widetilde{M}_{N;M}^{ba}(y_q,x_p) = \eta_q\,\frac{\hat{\mathcal{J}}_{N;M}^{ba}(y_q,x_p)}{\mathcal{N}}\,\eta_p+ \\ & +\;\;\delta^{(4)}(y_q-x_p)\,\delta_{pq}\,\eta_q\bigg[\bigg(\begin{matrix}\big[\hat{\beta}(\hat{\gamma}^\mu\hat{\partial}_{p,\mu}-\imath\,\hat{\varepsilon}_p+\hat{m})\big]_{N;M} & \hat{J}_{\psi\psi;N;M}(x_p) \\ \hat{J}_{\psi\psi;N;M}^{\dagger}(x_p) & -\big[\hat{\beta}(\hat{\gamma}^\mu\hat{\partial}_{p,\mu}-\imath\,\hat{\varepsilon}_p+\hat{m})\big]_{N;M}^{T}\end{matrix}\bigg)_{N;M}^{ba}+ \\ & +\bigg(\Big[\Big(\hat{\partial}_p^{\lambda}\hat{\mathfrak{S}}_{\gamma;\nu\lambda}^{(\hat{F})}(x_p)\Big)-s_\gamma(x_p)\,n_\nu\Big]\Big[-\imath\,\hat{\mathfrak{e}}_p^{(\hat{F})}+C_{\alpha\beta'\gamma'}\;\hat{\mathfrak{S}}_{\alpha}^{(\hat{F})\mu'\nu'}(x_p)\Big]_{\gamma\beta}^{-\mathbf{1};\nu\mu}+\frac{1}{2}\sum_{(\alpha)=1,..,8}^{(\kappa)=0,..,3}\hat{\mathfrak{B}}_{\hat{F};\beta(\alpha)}^{\mu(\kappa)}(x_p)\;\sigma_D^{(\alpha;\kappa)}(x_p)\bigg)_{\beta}^{\mu}\times \\ & \times\bigg(\begin{matrix}\big[\hat{\beta}(\imath\,\hat{\gamma}_\mu\,\hat{t}_\beta)\big]_{\overline{N};\overline{M}} & 0 \\ 0 & -\big[\hat{\beta}(\imath\,\hat{\gamma}_\mu\,\hat{t}_\beta)\big]_{\overline{N};\overline{M}}^{T}\end{matrix}\bigg)^{ba}\delta_{g,f}+\frac{1}{2}\hat{S}^{bb}\bigg(\begin{matrix}\delta\hat{\Sigma}_{N;M}^{11}(x_p) & \delta\hat{\Sigma}_{N;M}^{12}(x_p) \\ \delta\hat{\Sigma}_{N;M}^{21}(x_p) & -\delta\hat{\Sigma}_{N;M}^{22}(x_p)\end{matrix}\bigg)^{ba}\hat{S}^{aa}\bigg)\bigg]_{g,\overline{N};f,\overline{M}}^{ba}. \end{aligned} \tag{3.47}$$

According to Ref. [11], we transform the matrix $\widetilde{M}_{N;M}^{ba}(y_q,x_p)$ (3.47) with hermitian, BCS terms to $\widetilde{\mathcal{M}}_{N;M}^{ba}(y_q,x_p)$ (3.51) with anti-hermitian, anomalous parts by the subsequent steps (3.48-3.50); additional use is made of the delta-function

(3.43) for abbreviating the sum of the 32, colour dressed self-energies $\delta\widetilde{\Sigma}_{M;N}^{(\alpha;\kappa)ab}(x_p)$ by the single, anomalous doubled one $\delta\widetilde{\Sigma}_{M;N}^{ab}(x_p)$ (3.44)

$$\widetilde{M}_{N;M}^{ba}(y_q,x_p) \;\;\rightarrow\;\; \hat{S}\left(\hat{S}\,\widetilde{M}_{N;M}^{ba}(y_q,x_p)\,\hat{S}\right)\hat{S} \;\rightarrow\; \hat{S}\,\hat{I}^{-1}\Big(\underbrace{\hat{I}\,\hat{S}\,\widetilde{M}_{N;M}^{ba}(y_q,x_p)\,\hat{S}\,\hat{I}}_{\widetilde{\mathcal{M}}_{N;M}^{ba}(y_q,x_p)}\Big)\hat{I}^{-1}\,\hat{S}\,; \tag{3.48}$$

$$\begin{aligned}\widetilde{\mathcal{M}}_{N;M}^{ba}(y_q,x_p) &= \hat{I}\,\hat{S}\,\widetilde{M}_{N;M}^{ba}(y_q,x_p)\,\hat{S}\,\hat{I}\,;\\ \mathrm{DET}\Big\{\widetilde{M}_{N;M}^{ba}(y_q,x_p)\Big\} &= \mathrm{DET}\Big\{\hat{S}\,\hat{I}^{-1}\,\widetilde{\mathcal{M}}_{N;M}^{ba}(y_q,x_p)\,\hat{I}^{-1}\,\hat{S}\Big\} = \mathrm{DET}\Big\{\widetilde{\mathcal{M}}_{N;M}^{ba}(y_q,x_p)\Big\}\,; \qquad (3.49)\\ \widetilde{M}_{N;M}^{\boldsymbol{-1};ba}(y_q,x_p) &= \hat{S}\,\hat{I}\,\widetilde{\mathcal{M}}_{N;M}^{\boldsymbol{-1};ba}(y_q,x_p)\,\hat{I}\,\hat{S}\,; \qquad (3.50)\end{aligned}$$

$$\begin{aligned}\widetilde{\mathcal{M}}_{N;M}^{ba}(y_q,x_p) &= \hat{I}\,\hat{S}\,\eta_q\,\frac{\hat{\mathscr{J}}_{N;M}^{ba}(y_q,x_p)}{\mathcal{N}}\,\eta_p\,\hat{S}\,\hat{I} + \delta^{(4)}(y_q-x_p)\,\eta_q\,\delta_{pq}\,\times \qquad (3.51)\\ &\times\Bigg[\begin{pmatrix}\big[\hat{\beta}(\hat{\gamma}^{\mu}\hat{\partial}_{p,\mu}-\imath\,\hat{\varepsilon}_p+\hat{m})\big]_{N;M} & -\imath\,\hat{j}_{\psi\psi;N;M}(x_p)\\ -\imath\,\hat{j}_{\psi\psi;N;M}^{\dagger}(x_p) & \big[\hat{\beta}(\hat{\gamma}^{\mu}\hat{\partial}_{p,\mu}-\imath\,\hat{\varepsilon}_p+\hat{m})\big]_{N;M}^{T}\end{pmatrix}_{N;M}^{ba} +\\ &+\;\mathcal{V}_{\beta}^{\mu}(x_p)\begin{pmatrix}\big[\hat{\beta}(\imath\,\hat{\gamma}_{\mu}\,\hat{t}_{\beta})\big]_{\overline{N};\overline{M}} & 0\\ 0 & \big[\hat{\beta}(\imath\,\hat{\gamma}_{\mu}\,\hat{t}_{\beta})\big]_{\overline{N};\overline{M}}^{T}\end{pmatrix}_{g,\overline{N};f,\overline{M}}^{ba}\delta_{g,f} + \frac{1}{2}\begin{pmatrix}\delta\hat{\Sigma}_{N;M}^{11}(x_p) & \imath\,\delta\hat{\Sigma}_{N;M}^{12}(x_p)\\ \imath\,\delta\hat{\Sigma}_{N;M}^{21}(x_p) & \delta\hat{\Sigma}_{N;M}^{22}(x_p)\end{pmatrix}_{N;M}^{ba}\Bigg].\end{aligned}$$

Furthermore, we have shortened expressions for $\widetilde{\mathcal{M}}_{N;M}^{ba}(y_q,x_p)$ in (3.51) by substituting the field strength self-energy term of $\hat{\mathfrak{S}}_{\alpha;\mu\nu}^{(\hat{F})}(x_p)$ and the quark self-energy density terms of $\sigma_D^{(\alpha;\kappa)}(x_p)$ with $\hat{\mathfrak{B}}_{\hat{F};\beta\alpha}^{\mu\kappa}(x_p)$ by a potential $\mathcal{V}_{\beta}^{\mu}(x_p)$ (3.52). This real-valued potential variable $\mathcal{V}_{\beta}^{\mu}(x_p)$ (3.52) follows from the background path integral (3.42) with additional averaging of the quark self-energy densities

$$\mathcal{V}_{\beta}^{\mu}(x_p) = \Big[\Big(\hat{\partial}_p^{\lambda}\hat{\mathfrak{S}}_{\gamma;\nu\lambda}^{(\hat{F})}(x_p)\Big) - s_{\gamma}(x_p)\,n_{\nu}\Big]\,\Big[-\imath\,\hat{\mathfrak{e}}_p^{(\hat{F})} + C_{\alpha\beta'\gamma'}\,\hat{\mathfrak{S}}_{\alpha}^{(\hat{F})\mu'\nu'}(x_p)\Big]_{\gamma\beta}^{\boldsymbol{-1};\nu\mu} + \frac{1}{2}\sum_{(\alpha)=1,..,8}^{(\kappa)=0,..,3}\hat{\mathfrak{B}}_{\hat{F};\beta(\alpha)}^{\mu(\kappa)}(x_p)\;\;\sigma_D^{(\alpha;\kappa)}(x_p)\,. \tag{3.52}$$

Eventually, the matrix $\widetilde{M}_{N;M}^{ba}(y_q,x_p)$ is replaced by $\widetilde{\mathcal{M}}_{N;M}^{ba}(y_q,x_p)$ (3.51,3.52) in (3.46) so that we achieve the generating function $Z[\hat{\mathscr{J}},J_{\psi},\hat{J}_{\psi\psi},\hat{\mathfrak{j}}^{(\hat{F})}]$ (3.53,3.54) which is further reduced by shifting the anti-hermitian, anomalous doubled, single self-energy (3.44) in $\widetilde{\mathcal{M}}_{N;M}^{ba}(y_q,x_p)$ with the source matrix $\imath\,\hat{J}_{\psi\psi;N;M}^{b\neq a}(x_p)$ (3.55). In consequence the source matrix $\imath\,\hat{J}_{\psi\psi;N;M}^{b\neq a}(x_p)$ disappears from $\widetilde{\mathcal{M}}_{N;M}^{ba}(y_q,x_p)$ (3.51), but modifies the delta-function (3.43) into (3.56). The background path integral of gauge field degrees of freedom is listed again in relation (3.54) for convenience, but additionally includes averaging with the real, scalar quark self-energy densities $d[\sigma_D^{(\alpha;\kappa)}(x_p)]$ (boldface symbols in (3.54), last line in Eq. (3.54))

$$\begin{aligned}Z[\hat{\mathscr{J}},J_{\psi},\hat{J}_{\psi\psi},\hat{j}_{\hat{F}}] &= \Big\langle \boldsymbol{Z}\Big[\boldsymbol{\hat{\mathfrak{S}}^{(\hat{F})}, s_{\alpha}, \hat{\mathfrak{B}}_{\hat{F}}, \hat{\mathfrak{b}}^{(\hat{F})}, \hat{\mathfrak{U}}_{\hat{F}}, \hat{\mathfrak{v}}_{\hat{F}}; \sigma_D;; \hat{\mathfrak{j}}^{(\hat{F})}; \mathrm{Eq.\ (3.54)}}\Big]\times \qquad (3.53)\\ &\times\int d[\delta\widetilde{\Sigma}_{M;N}^{(\alpha;\kappa)ab}(x_p)]\;\exp\Big\{\frac{\imath}{8}\int_C d^4x_p\sum_{(\alpha)=1,..,8}^{(\kappa)=0,..,3}\Big[-\imath\,\hat{\mathfrak{e}}_p^{(\hat{F})}+\hat{\mathfrak{b}}_{(\alpha;\kappa)}^{(\hat{F})}(x_p)\Big]\times\\ &\times\underset{N_f,\hat{\gamma}_{mn}^{(\mu)},N_c}{\overset{a(=1,2)}{\mathfrak{Tr}}}\Bigg[\delta\widetilde{\Sigma}_{f,\overline{M};g,\overline{N}}^{(\alpha;\kappa)ab}(x_p)\,\frac{1}{\hat{\mathfrak{v}}_{\hat{F};\overline{N}}^{(\alpha;\kappa)}(x_p)}\,\delta\widetilde{\Sigma}_{g,\overline{N};f,\overline{M}}^{(\alpha;\kappa)ba}(x_p)\,\frac{1}{\hat{\mathfrak{v}}_{\hat{F};\overline{M}}^{(\alpha;\kappa)}(x_p)}\Bigg]\Bigg\}\times\int d[\delta\widetilde{\Sigma}_{M;N}^{ab}(x_p)]\;\times\\ &\times\;\delta\Big(\delta\widetilde{\Sigma}_{f,\overline{M};g,\overline{N}}^{ab}(x_p)+2\,\imath\,\hat{J}_{\psi\psi;f,\overline{M};g,\overline{N}}^{a\neq b}(x_p)-\sum_{(\alpha)=1,..,8}^{(\kappa)=0,..,3}\hat{\mathfrak{U}}_{\hat{F};\overline{M};\overline{M}'}^{(\alpha;\kappa),aa,\dagger}(x_p)\,\delta\widetilde{\Sigma}_{f,\overline{M}';g,\overline{N}'}^{(\alpha;\kappa)ab}(x_p)\,\hat{\mathfrak{U}}_{\hat{F};\overline{N}';\overline{N}}^{(\alpha;\kappa),bb}(x_p)\Big)\times\\ &\times\;\Big\{\mathrm{DET}\Big[\widetilde{\mathcal{M}}_{N;M}^{ba}(y_q,x_p)\Big]\Big\}^{1/2}\;\exp\Big\{\frac{\imath}{2}\int_C d^4x_p\,d^4y_q\;\;J_{\psi;N}^{\dagger,b}(y_q)\,\hat{I}\,\widetilde{\mathcal{M}}_{N;M}^{\boldsymbol{-1};ba}(y_q,x_p)\,\hat{I}\,J_{\psi;M}^{a}(x_p)\Big\}\Big\rangle\,;\end{aligned}$$

$$\Big\langle \boldsymbol{Z}\Big[\hat{\boldsymbol{\mathfrak{S}}}^{(\hat{F})},\boldsymbol{s_\alpha},\hat{\boldsymbol{\mathfrak{B}}}_{\hat{F}},\hat{\boldsymbol{\mathfrak{b}}}^{(\hat{F})},\hat{\boldsymbol{\mathfrak{U}}}_{\hat{F}},\hat{\boldsymbol{\mathfrak{v}}}_{\hat{F}};\boldsymbol{\sigma_D};\hat{\boldsymbol{\mathfrak{j}}}^{(\hat{F})};\textbf{Eq. (3.54)}\Big]\Big(\textbf{fields}\Big)\Big\rangle = \tag{3.54}$$
$$= \Big\langle \int d[\hat{\mathfrak{S}}_{\alpha;\mu\nu}^{(\hat{F})}(x_p)]\;\; d[\hat{\mathfrak{B}}_{\hat{F};\beta(\alpha)}^{\mu(\kappa)}(x_p);\hat{\mathfrak{b}}_{(\alpha;\kappa)}^{(\hat{F})}(x_p)]\;\; d[\hat{\mathfrak{U}}_{\hat{F};\overline{N};\overline{M}}^{(\alpha;\kappa)}(x_p);\hat{\mathfrak{v}}_{\hat{F};\overline{N}}^{(\alpha;\kappa)}(x_p)]\;\times$$
$$\times\;\Big\{\prod_{\{x_p\}}\delta\Big(C_{\alpha\beta\gamma}\;\hat{\mathfrak{S}}_\alpha^{(\hat{F})\mu\nu}(x_p)-\sum_{(\alpha)=1,..,8}^{(\kappa)=0,..,3}\hat{\mathfrak{B}}_{\hat{F};\beta(\alpha)}^{\mu(\kappa)}(x_p)\;\hat{\mathfrak{b}}_{(\alpha;\kappa)}^{(\hat{F})}(x_p)\;\hat{\mathfrak{B}}_{\hat{F};(\alpha)\gamma}^{T,(\kappa)\nu}(x_p)\Big)\Big\}\times$$
$$\times\;\Big\{\prod_{\{x_p\};(\alpha)=1,..,8}^{\{(\kappa)=0,..,3\}}\delta\Big(\hat{\mathfrak{U}}_{\hat{F};\overline{M}';\overline{N}}^{(\alpha;\kappa),11,\dagger}(x_p)\;\hat{\mathfrak{v}}_{\hat{F};\overline{N}}^{(\alpha;\kappa)}(x_p)\;\hat{\mathfrak{U}}_{\hat{F};\overline{N};\overline{M}}^{(\alpha;\kappa),11}(x_p)-\big[\hat{\beta}\,(\imath\;\hat{\gamma}_\mu\;\hat{t}_\beta)\big]_{\overline{M}';\overline{M}}\;\hat{\mathfrak{B}}_{\hat{F};\beta(\alpha)}^{\mu(\kappa)}(x_p)\Big)\Big\}\times$$
$$\times\;\mathfrak{P}_1\Big(\hat{\mathfrak{b}}_{(\alpha;\kappa)}^{(\hat{F})}(x_p)\Big)\;\mathfrak{P}_2\Big(\hat{\mathfrak{b}}_{(\alpha;\kappa)}^{(\hat{F})}(x_p)\Big)\;\mathfrak{P}_3\Big(\hat{\mathfrak{b}}_{(\alpha;\kappa)}^{(\hat{F})}(x_p)\Big)\;\;\mathfrak{Q}_1\Big(\hat{\mathfrak{v}}_{\hat{F};\overline{M}}^{(\alpha;\kappa)}(x_p)\Big)\;\mathfrak{Q}_3\Big(\hat{\mathfrak{v}}_{\hat{F};\overline{M}}^{(\alpha;\kappa)}(x_p)\Big)\;\int d[s_\alpha(x_p)]\;\times$$
$$\times\;\exp\Big\{\imath\int_C d^4x_p\Big(\frac{1}{4}\hat{\mathfrak{S}}_\alpha^{(\hat{F})\mu\nu}(x_p)\;\hat{\mathfrak{S}}_{\alpha;\mu\nu}^{(\hat{F})}(x_p)-\hat{\mathfrak{j}}_{\alpha;\mu\nu}^{(\hat{F})}(x_p)\;\hat{\mathfrak{S}}_\alpha^{(\hat{F})\mu\nu}(x_p)+\hat{\mathfrak{j}}_{\alpha;\mu\nu}^{(\hat{F})}(x_p)\;\hat{\mathfrak{j}}_\alpha^{(\hat{F})\mu\nu}(x_p)\Big)\Big\}\times$$
$$\times\;\Big\{\det\Big[\Big(-\imath\,\hat{\mathfrak{e}}_p^{(\hat{F})}+C_{\alpha\beta\gamma}\;\hat{\mathfrak{S}}_\alpha^{(\hat{F})\mu\nu}(x_p)\Big)_{\beta\gamma}^{\mu\nu}\Big]\Big\}^{\boldsymbol{-1/2}}\times\exp\Big\{\frac{\imath}{2}\int_C d^4x_p\;\Big[\big(\hat{\partial}_p^\lambda\hat{\mathfrak{S}}_{\gamma;\nu\lambda}^{(\hat{F})}(x_p)\big)-s_\gamma(x_p)\;n_\nu\Big]\;\times$$
$$\times\;\Big[-\imath\,\hat{\mathfrak{e}}_p^{(\hat{F})}+C_{\alpha\beta'\gamma'}\;\hat{\mathfrak{S}}_\alpha^{(\hat{F})\mu'\nu'}(x_p)\Big]_{\gamma\beta}^{\boldsymbol{-1};\nu\mu}\;\Big[\big(\hat{\partial}_p^\kappa\hat{\mathfrak{S}}_{\beta;\mu\kappa}^{(\hat{F})}(x_p)\big)-s_\beta(x_p)\;n_\mu\Big]\Big\}\;\times$$
$$\times\;\int \boldsymbol{d[\sigma_D^{(\alpha;\kappa)}(x_p)]}\;\;\boldsymbol{\exp\Big\{\frac{\imath}{4}\int_C d^4x_p\sum_{(\alpha)=1,..,8}^{(\kappa)=0,..,3}\Big[\imath\,\hat{\mathfrak{e}}_p^{(\hat{F})}+\hat{\mathfrak{b}}_{(\alpha;\kappa)}^{(\hat{F})}(x_p)\Big]\Big(\sigma_D^{(\alpha;\kappa)}(x_p)\Big)^2\Big\}}\times\Big(\textbf{fields}\Big)\Big\rangle\;;$$

$$\frac{1}{2}\begin{pmatrix}\delta\hat{\Sigma}_{N;M}^{11}(x_p) & \imath\,\delta\hat{\Sigma}_{N;M}^{12}(x_p)\\ \imath\,\delta\hat{\Sigma}_{N;M}^{21}(x_p) & \delta\hat{\Sigma}_{N;M}^{22}(x_p)\end{pmatrix}-\begin{pmatrix}0 & \imath\;\hat{j}_{\psi\psi;N;M}(x_p)\\ \imath\;\hat{j}_{\psi\psi;N;M}^{\dagger}(x_p) & 0\end{pmatrix}\rightarrow\frac{1}{2}\begin{pmatrix}\delta\hat{\Sigma}_{N;M}^{11}(x_p) & \imath\,\delta\hat{\Sigma}_{N;M}^{12}(x_p)\\ \imath\,\delta\hat{\Sigma}_{N;M}^{21}(x_p) & \delta\hat{\Sigma}_{N;M}^{22}(x_p)\end{pmatrix}; \tag{3.55}$$

$$\delta\Big(\delta\widetilde{\Sigma}_{f,\overline{M};g,\overline{N}}^{ab}(x_p)-\sum_{(\alpha)=1,..,8}^{(\kappa)=1,..,3}\hat{\mathfrak{U}}_{\hat{F};\overline{M};\overline{M}'}^{(\alpha;\kappa),aa,\dagger}(x_p)\;\;\delta\widetilde{\Sigma}_{f,\overline{M}';g,\overline{N}'}^{(\alpha;\kappa)ab}(x_p)\;\;\hat{\mathfrak{U}}_{\hat{F};\overline{N}';\overline{N}}^{(\alpha;\kappa),bb}(x_p)\Big)\;\rightarrow \tag{3.56}$$
$$\rightarrow\;\;\delta\Big(\delta\widetilde{\Sigma}_{f,\overline{M};g,\overline{N}}^{ab}(x_p)+2\,\imath\;\hat{J}_{\psi\psi;f,\overline{M};g,\overline{N}}^{a\neq b}(x_p)-\sum_{(\alpha)=1,..,8}^{(\kappa)=0,..,3}\hat{\mathfrak{U}}_{\hat{F};\overline{M};\overline{M}'}^{(\alpha;\kappa),aa,\dagger}(x_p)\;\delta\widetilde{\Sigma}_{f,\overline{M}';g,\overline{N}'}^{(\alpha;\kappa)ab}(x_p)\;\hat{\mathfrak{U}}_{\hat{F};\overline{N}';\overline{N}}^{(\alpha;\kappa),bb}(x_p)\Big)\,.$$

Furthermore, we move the potential variable $\mathcal{V}_\beta^\mu(x_p)$ (3.52), which consists of the field strength self-energy $\hat{\mathfrak{S}}_{\alpha;\mu\nu}^{(\hat{F})}(x_p)$, other colour-related degrees of freedom as $\hat{\mathfrak{B}}_{\hat{F};\beta(\alpha)}^{\mu(\kappa)}(x_p)$ and the quark self-energy densities $\sigma_D^{(\alpha;\kappa)}(x_p)$, to the background functional (3.54). This transformation becomes obvious, as one expands the anomalous doubled determinant $\{\mathrm{DET}[\widetilde{\mathcal{M}}_{N;M}^{ba}(y_q,x_p)\,]\}^{1/2}$ and Green function $\widetilde{\mathcal{M}}_{N;M}^{\boldsymbol{-1};ba}(y_q,x_p)$ in (3.53,3.51) in terms of $\hat{H}_{N;M}(x_p)=[\hat{\beta}(\,\hat{\not{\partial}}_p+\imath\,\hat{\not{\mathcal{V}}}(x_p)-\imath\,\hat{\varepsilon}_p+\hat{m})]_{N;M}$ (boldface symbols in Eqs. (3.57-3.59)). Since the part $\hat{H}_{N;M}(x_p)=[\hat{\beta}(\,\hat{\not{\partial}}_p+\imath\,\hat{\not{\mathcal{V}}}(x_p)-\imath\,\hat{\varepsilon}_p+\hat{m})]_{N;M}$ is contained in $\widetilde{\mathcal{M}}_{N;M}^{ba}(y_q,x_p)$ in its anomalous doubled kind with the transpose $\hat{H}_{N;M}^T(x_p)$, one can simplify the separated actions for the background functionals by multiplying with an additional factor of two. The particular form of the anomalous doubled Hilbert space is described in appendix A, which also summarizes the definitions of the doubled spacetime coordinate states and the appropriate decomposition of the unit operator into complete sets of anomalous doubled spacetime or momentum-energy states. The decomposition of the unit operator also yields the suitable form of trace operations of the doubled Hilbert space with inclusion of the time contour integrals for forward and backward propagation (compare also with chapter 4 in [11])

$$\Big\{\mathrm{DET}[\widetilde{\mathcal{M}}_{N;M}^{ba}(y_q,x_p)\,]\Big\}^{1/2}=\exp\Big\{\frac{1}{2}\underset{\int_C d^4x_p\;\eta_p}{\overset{a(=1,2)}{\mathrm{TR}}}\Big[\underset{N_f,\hat{\gamma}_{mn}^{(\mu)},N_c}{\mathfrak{tr}}\;\ln\Big[\widetilde{\mathcal{M}}_{N;M}^{ba}(y_q,x_p)\Big]\Big]\Big\}= \tag{3.57}$$

$$= \exp\Bigg\{\frac{1}{2}\;\mathop{\mathrm{TR}}_{\int_C d^4x_p\;\eta_p}^{a(=1,2)}\Bigg[\mathop{\mathfrak{tr}}_{N_f,\hat{\gamma}_{mn}^{(\mu)},N_c}\Bigg(\ln\Big[\widetilde{\mathcal{M}}_{N;M}^{ba}(y_q,x_p)\Big]-\ln\begin{bmatrix}\hat{\mathcal{H}}_{N;M}^{11}(y_q,x_p) & 0\\ 0 & \hat{\mathcal{H}}_{N;M}^{11,T}(y_q,x_p)\end{bmatrix}\Bigg)\Bigg]\Bigg\}\times$$

$$\times\;\underbrace{\boldsymbol{\exp}\Bigg\{\mathop{\mathrm{tr}}_{\int_C \boldsymbol{d^4x_p\;\eta_p}}\Bigg[\mathop{\boldsymbol{\mathfrak{tr}}}_{\boldsymbol{N_f,\hat{\gamma}_{mn}^{(\mu)},N_c}}\boldsymbol{\ln}\Big[\boldsymbol{\eta_p\hat{\beta}\Big(\hat{\partial}_p+\imath\,\hat{\mathcal{V}}(x_p)-\imath\,\hat{\varepsilon}_p+\hat{m}\Big)}\Big]\Bigg]\Bigg\}}_{\text{Move to background functional (3.54)} \to \text{(3.59)}}\;;$$

$$\exp\Bigg\{\frac{\imath}{2}\int_C d^4x_p\;d^4y_q\;\;J_{\psi;N}^{\dagger,b}(y_q)\;\hat{I}\;\widetilde{\mathcal{M}}_{N;M}^{-\boldsymbol{1};ba}(y_q,x_p)\;\hat{I}\;J_{\psi;M}^{a}(x_p)\Bigg\}=\exp\Bigg\{\frac{\imath}{2}\int_C d^4x_p\;d^4y_q\;\times \tag{3.58}$$

$$\times\;\;J_{\psi;N}^{\dagger,b}(y_q)\;\hat{I}\;\Bigg(\Bigg(\widetilde{\mathcal{M}}_{N;M}^{-\boldsymbol{1};ba}(y_q,x_p)-\begin{pmatrix}\hat{\mathcal{H}}_{N;M}^{11;-\boldsymbol{1}}(y_q,x_p) & 0\\ 0 & \hat{\mathcal{H}}_{N;M}^{11,T;-\boldsymbol{1}}(y_q,x_p)\end{pmatrix}\Bigg)\Bigg)\hat{I}\;J_{\psi;M}^{a}(x_p)\Bigg\}\times$$

$$\times\;\;\underbrace{\boldsymbol{\exp\Bigg\{\imath\int_C d^4x_p\;d^4y_q\;\;j_{\psi;N}^{\dagger}(y_q)\;\Big\langle y_q\Big|\Big[\hat{\beta}\big(\hat{\partial}_p+\imath\,\hat{\mathcal{V}}(x_p)-\imath\,\hat{\varepsilon}_p+\hat{m}\big)\Big]_{N;M}^{-1}\Big|x_p\Big\rangle\;j_{\psi;M}(x_p)\Bigg\}}}_{\text{Move to background functional (3.54)} \to \text{(3.59)}}\;;$$

$$\Bigg\langle \boldsymbol{Z\Big[\hat{\mathcal{V}}(x_p);\hat{\mathfrak{S}}^{(\hat{F})},s_\alpha,\hat{\mathfrak{B}}_{\hat{F}},\hat{\mathfrak{b}}^{(\hat{F})},\hat{\mathfrak{U}}_{\hat{F}},\hat{\mathfrak{v}}_{\hat{F}};\sigma_D;\hat{\mathfrak{j}}^{(\hat{F})};\textbf{Eq. (3.59)}\Big]\;\Big(\textbf{fields}\Big)}\Bigg\rangle= \tag{3.59}$$

$$=\;\;\Bigg\langle\int d[\hat{\mathfrak{S}}_\alpha^{(\hat{F})\mu\nu}(x_p)]\;\;d[\hat{\mathfrak{B}}_{\hat{F};\beta(\alpha)}^{\mu(\kappa)}(x_p);\hat{\mathfrak{b}}_{(\alpha;\kappa)}^{(\hat{F})}(x_p)]\;\;d[\hat{\mathfrak{U}}_{\hat{F};\overline{N};\overline{M}}^{(\alpha;\kappa)}(x_p);\hat{\mathfrak{v}}_{\hat{F};\overline{N}}^{(\alpha;\kappa)}(x_p)]\;\times$$

$$\times\;\;\Bigg\{\prod_{\{x_p\}}\delta\Bigg(C_{\alpha\beta\gamma}\;\hat{\mathfrak{S}}_\alpha^{(\hat{F})\mu\nu}(x_p)-\sum_{(\alpha)=1,..,8}^{(\kappa)=0,..,3}\hat{\mathfrak{B}}_{\hat{F};\beta(\alpha)}^{\mu(\kappa)}(x_p)\;\hat{\mathfrak{b}}_{(\alpha;\kappa)}^{(\hat{F})}(x_p)\;\hat{\mathfrak{B}}_{\hat{F};(\alpha)\gamma}^{T,(\kappa)\nu}(x_p)\Bigg)\Bigg\}\times$$

$$\times\;\;\Bigg\{\prod_{\{x_p\};(\alpha)=1,..,8}^{\{(\kappa)=0,..,3\}}\delta\Bigg(\hat{\mathfrak{U}}_{\hat{F};\overline{M}';\overline{N}}^{(\alpha;\kappa),11,\dagger}(x_p)\;\hat{\mathfrak{v}}_{\hat{F};\overline{N}}^{(\alpha;\kappa)}(x_p)\;\hat{\mathfrak{U}}_{\hat{F};\overline{N};\overline{M}}^{(\alpha;\kappa),11}(x_p)-\big[\hat{\beta}\,(\imath\;\hat{\gamma}_\mu\;\hat{t}_\beta)\big]_{\overline{M}';\overline{M}}\;\hat{\mathfrak{B}}_{\hat{F};\beta(\alpha)}^{\mu(\kappa)}(x_p)\Bigg)\Bigg\}\times$$

$$\times\;\;\mathfrak{P}_1\Big(\hat{\mathfrak{b}}_{(\alpha;\kappa)}^{(\hat{F})}(x_p)\Big)\;\mathfrak{P}_2\Big(\hat{\mathfrak{b}}_{(\alpha;\kappa)}^{(\hat{F})}(x_p)\Big)\;\mathfrak{P}_3\Big(\hat{\mathfrak{b}}_{(\alpha;\kappa)}^{(\hat{F})}(x_p)\Big)\;\;\mathfrak{Q}_1\Big(\hat{\mathfrak{v}}_{\hat{F};\overline{M}}^{(\alpha;\kappa)}(x_p)\Big)\;\mathfrak{Q}_3\Big(\hat{\mathfrak{v}}_{\hat{F};\overline{M}}^{(\alpha;\kappa)}(x_p)\Big)\;\int d[s_\alpha(x_p)]\;\times$$

$$\times\;\;\exp\Bigg\{\imath\int_C d^4x_p\Bigg(\frac{1}{4}\hat{\mathfrak{S}}_\alpha^{(\hat{F})\mu\nu}(x_p)\;\hat{\mathfrak{S}}_{\alpha;\mu\nu}^{(\hat{F})}(x_p)-\hat{\mathfrak{j}}_{\alpha;\mu\nu}^{(\hat{F})}(x_p)\;\hat{\mathfrak{S}}_\alpha^{(\hat{F})\mu\nu}(x_p)+\hat{\mathfrak{j}}_{\alpha;\mu\nu}^{(\hat{F})}(x_p)\;\hat{\mathfrak{j}}_\alpha^{(\hat{F})\mu\nu}(x_p)\Bigg)\Bigg\}\times$$

$$\times\;\;\Bigg\{\det\Bigg[\Big(-\imath\,\hat{\mathfrak{e}}_p^{(\hat{F})}+C_{\alpha\beta\gamma}\;\hat{\mathfrak{S}}_\alpha^{(\hat{F})\mu\nu}(x_p)\Big)_{\beta\gamma}^{\mu\nu}\Bigg]\Bigg\}^{-\boldsymbol{1/2}}\times\exp\Bigg\{\frac{\imath}{2}\int_C d^4x_p\;\Big[\big(\,\hat{\partial}_p^\lambda\hat{\mathfrak{S}}_{\gamma;\nu\lambda}^{(\hat{F})}(x_p)\,\big)-s_\gamma(x_p)\;n_\nu\Big]\;\times$$

$$\times\;\;\Big[-\imath\,\hat{\mathfrak{e}}_p^{(\hat{F})}+C_{\alpha\beta'\gamma'}\;\hat{\mathfrak{S}}_\alpha^{(\hat{F})\mu'\nu'}(x_p)\Big]_{\gamma\beta}^{-\boldsymbol{1};\nu\mu}\;\Big[\big(\,\hat{\partial}_p^\kappa\hat{\mathfrak{S}}_{\beta;\mu\kappa}^{(\hat{F})}(x_p)\,\big)-s_\beta(x_p)\;n_\mu\Big]\Bigg\}\;\times$$

$$\times\;\;\int d[\sigma_D^{(\alpha;\kappa)}(x_p)]\;\exp\Bigg\{\frac{\imath}{4}\int_C d^4x_p\sum_{(\alpha)=1,..,8}^{(\kappa)=0,..,3}\Big[\imath\,\hat{\mathfrak{e}}_p^{(\hat{F})}+\hat{\mathfrak{b}}_{(\alpha;\kappa)}^{(\hat{F})}(x_p)\Big]\;\Big(\sigma_D^{(\alpha;\kappa)}(x_p)\Big)^2\Bigg\}\times$$

$$\times\;\;\boldsymbol{\exp}\Bigg\{\mathop{\mathrm{tr}}_{\int_C \boldsymbol{d^4x_p\;\eta_p}}\Bigg[\mathop{\boldsymbol{\mathfrak{tr}}}_{\boldsymbol{N_f,\hat{\gamma}_{mn}^{(\mu)},N_c}}\boldsymbol{\ln}\Big[\boldsymbol{\eta_p\hat{\beta}\Big(\hat{\partial}_p+\imath\,\hat{\mathcal{V}}(x_p)-\imath\,\hat{\varepsilon}_p+\hat{m}\Big)}\Big]\Bigg]\Bigg\}\times$$

$$\times\;\;\boldsymbol{\exp\Bigg\{\imath\int_C d^4x_p\;d^4y_q\;\;j_{\psi;N}^{\dagger}(y_q)\;\Big\langle y_q\Big|\Big[\hat{\beta}\big(\hat{\partial}_p+\imath\,\hat{\mathcal{V}}(x_p)-\imath\,\hat{\varepsilon}_p+\hat{m}\big)\Big]_{N;M}^{-1}\Big|x_p\Big\rangle\;j_{\psi;M}(x_p)\Bigg\}\Big(\textbf{fields}\Big)}\Bigg\rangle\;;$$

$$\begin{aligned} \hat{\mathcal{H}}^{11}_{N;M}(y_q,x_p) &= \delta_{pq}\;\delta^{(4)}(y_q-x_p)\;\eta_q\Big[\hat{\beta}\big(\hat{\not\partial}_p+\imath\,\hat{\not\mathcal{V}}(x_p)-\imath\,\hat{\varepsilon}_p+\hat{m}\big)\Big]_{N;M}\;; \\ \mathcal{V}^{\mu}_{\beta}(x_p) &= \Big[\Big(\hat{\partial}^{\lambda}_p\hat{\mathfrak{S}}^{(\hat{F})}_{\gamma;\nu\lambda}(x_p)\Big)-s_{\gamma}(x_p)\;n_{\nu}\Big]\;\Big[-\imath\,\hat{\mathfrak{e}}^{(\hat{F})}_p+C_{\alpha\beta'\gamma'}\;\hat{\mathfrak{S}}^{(\hat{F})\mu'\nu'}_{\alpha}(x_p)\Big]^{-1;\nu\mu}_{\gamma\beta}+ \\ &+\;\frac{1}{2}\sum_{(\alpha)=1,..,8}^{(\kappa)=0,..,3}\hat{\mathfrak{B}}^{\mu(\kappa)}_{\hat{F};\beta(\alpha)}(x_p)\;\;\sigma^{(\alpha;\kappa)}_{D}(x_p)\;. \end{aligned} \tag{3.60}$$

In consistency to the shift of $\frac{1}{2}\;\delta\widetilde{\Sigma}^{ab}_{M;N}(x_p)$ by $\imath\;\hat{J}^{a\neq b}_{\psi\psi;M;N}(x_p)$ (3.55), one attains a simplified matrix $\widetilde{\mathcal{M}}^{ba}_{N;M}(y_q,x_p)$ (3.61) without $\imath\,\hat{J}^{a\neq b}_{\psi\psi;M;N}(x_p)$ instead of (3.51), but has a modified delta function (3.56) instead of (3.43) for the sum of 32, colour dressed self-energies $\delta\widetilde{\Sigma}^{(\alpha;\kappa)ab}_{M;N}(x_p)$. The new delta function (3.56) is reduced to an additional effective functional (3.62), which has a local spacetime dependence and which follows from integrating by $\delta\widetilde{\Sigma}^{(\alpha;\kappa)ab}_{M;N}(x_p)$ over the delta function (3.56) and over remaining Gaussian factors. These Gaussian factors consist of actions with the eigenvalues $\hat{\mathfrak{b}}^{(\hat{F})}_{(\alpha;\kappa)}(x_p)$, $\hat{\mathfrak{v}}^{(\alpha;\kappa)}_{\hat{F};\overline{N}}(x_p)$ as gauge field variables from the diagonalization of the interaction potentials (3.17,3.18) and (3.22)

$$\begin{aligned} \widetilde{\mathcal{M}}^{ba}_{N;M}(y_q,x_p) &= \hat{I}\;\hat{S}\;\eta_q\,\frac{\hat{\mathcal{J}}^{ba}_{N;M}(y_q,x_p)}{\mathcal{N}}\,\eta_p\;\hat{S}\;\hat{I}+\delta^{(4)}(y_q-x_p)\;\eta_q\;\delta_{pq}\;\times \\ \times\;&\Bigg[\begin{pmatrix}\big[\hat{\beta}\,(\hat{\not\partial}_p+\imath\,\hat{\not\mathcal{V}}(x_p)-\imath\,\hat{\varepsilon}_p+\hat{m})\big]_{N;M} & 0 \\ 0 & \big[\hat{\beta}\,(\hat{\not\partial}_p+\imath\,\hat{\not\mathcal{V}}(x_p)-\imath\,\hat{\varepsilon}_p+\hat{m})\big]^{T}_{N;M}\end{pmatrix}^{ba}_{N;M}+ \\ +\;\;&\frac{1}{2}\begin{pmatrix}\delta\hat{\Sigma}^{11}_{N;M}(x_p) & \imath\,\delta\hat{\Sigma}^{12}_{N;M}(x_p) \\ \imath\,\delta\hat{\Sigma}^{21}_{N;M}(x_p) & \delta\hat{\Sigma}^{22}_{N;M}(x_p)\end{pmatrix}^{ba}_{N;M}\Bigg]^{ba}_{N;M}\;; \end{aligned} \tag{3.61}$$

$$\begin{aligned} &\Bigg\{\prod_{\{x_p\}}\boldsymbol{\Delta}\Big(\delta\widetilde{\Sigma}^{ab}_{f,\overline{M};g,\overline{N}}(x_p)+2\,\imath\hat{J}^{a\neq b}_{\psi\psi;f,\overline{M};g,\overline{N}}(x_p)\;;\;\hat{\mathcal{U}}^{(\alpha;\kappa)}_{\hat{F};\overline{M};\overline{N}}(x_p)\;;\;\hat{\mathfrak{v}}^{(\alpha;\kappa)}_{\hat{F};\overline{N}}(x_p)\;;\;\hat{\mathfrak{b}}^{(\hat{F})}_{(\alpha;\kappa)}(x_p)\Big)\Bigg\}= \\ &=\int d[\delta\widetilde{\Sigma}^{(\alpha;\kappa)ab}_{M;N}(x_p)]\;\exp\Bigg\{\frac{\imath}{8}\int_C d^4x_p\sum_{(\alpha)=1,..,8}^{(\kappa)=0,..,3}\Big[-\imath\,\hat{\mathfrak{e}}^{(\hat{F})}_p+\hat{\mathfrak{b}}^{(\hat{F})}_{(\alpha;\kappa)}(x_p)\Big]\times \\ &\times\underset{N_f,\hat{\gamma}^{(\mu)}_{mn},N_c}{\overset{a(=1,2)}{\mathfrak{Tr}}}\Bigg[\delta\widetilde{\Sigma}^{(\alpha;\kappa)ab}_{f,\overline{M};g,\overline{N}}(x_p)\frac{1}{\hat{\mathfrak{v}}^{(\alpha;\kappa)}_{\hat{F};\overline{N}}(x_p)}\delta\widetilde{\Sigma}^{(\alpha;\kappa)ba}_{g,\overline{N};f,\overline{M}}(x_p)\frac{1}{\hat{\mathfrak{v}}^{(\alpha;\kappa)}_{\hat{F};\overline{M}}(x_p)}\Bigg]\Bigg\}\times \\ &\times\Bigg\{\prod_{\{x_p\}\}}\delta\Big(\delta\widetilde{\Sigma}^{ab}_{f,\overline{M};g,\overline{N}}(x_p)+2\,\imath\hat{J}^{a\neq b}_{\psi\psi;f,\overline{M};g,\overline{N}}(x_p)-\sum_{(\alpha)=1,..,8}^{(\kappa)=0,..,3}\hat{\mathcal{U}}^{(\alpha;\kappa),aa,\dagger}_{\hat{F};\overline{M};\overline{M}'}(x_p)\;\delta\widetilde{\Sigma}^{(\alpha;\kappa)ab}_{f,\overline{M}';g,\overline{N}'}(x_p)\;\hat{\mathcal{U}}^{(\alpha;\kappa),bb}_{\hat{F};\overline{N}';\overline{N}}(x_p)\Big)\Bigg\}. \end{aligned} \tag{3.62}$$

We can therefore remove the 32, colour dressed self-energies $\delta\widetilde{\Sigma}^{(\alpha;\kappa)ab}_{M;N}(x_p)$ by inserting the effective functional (3.62) into (3.53). Furthermore, it has to be considered that the background path integral (3.54) has been transformed by separating effective one-particle potentials $\hat{H}_{N;M}(x_p)$, $\hat{H}^{T}_{N;M}(x_p)$ with $\mathcal{V}^{\mu}_{\beta}(x_p)$ (3.60), also containing the quark self-energy densities $\sigma^{(\alpha;\kappa)}_{D}(x_p)$, to the new background generating function (3.59). These transformations require adjustment of the anomalous doubled determinant (3.57) and bilinear source term (3.58) with the simplified matrix $\widetilde{\mathcal{M}}^{ba}_{N;M}(y_q,x_p)$ (3.61). The transformations (3.55-3.58) finally yield the generating function (3.63) whose most important dependence is given by the single, anomalous doubled self-energy $\delta\widetilde{\Sigma}^{ab}_{M;N}(x_p)$ with anti-hermitian BCS terms. The transformations (3.55-3.58) cause a different background field averaging with the path integral (3.59,3.60) instead of (3.54)

$$\begin{aligned} Z[\hat{\mathcal{J}},J_{\psi},\hat{J}_{\psi\psi},\hat{\mathfrak{j}}^{(\hat{F})}] &= \Big\langle\boldsymbol{Z\Big[\hat{\not\mathcal{V}}(x_p);\hat{\mathfrak{S}}^{(\hat{F})},s_{\alpha},\hat{\mathfrak{B}}_{\hat{F}},\hat{\mathfrak{b}}^{(\hat{F})},\hat{\mathcal{U}}_{\hat{F}},\hat{\mathfrak{v}}_{\hat{F}};\sigma_D;\hat{\mathfrak{j}}^{(\hat{F})};\textbf{Eq. (3.59)}\Big]}\times\int d[\delta\widetilde{\Sigma}^{ab}_{M;N}(x_p)]\times \\ \times\;\;&\Bigg\{\prod_{\{x_p\}}\boldsymbol{\Delta}\Big(\delta\widetilde{\Sigma}^{ab}_{f,\overline{M};g,\overline{N}}(x_p)+2\,\imath\hat{J}^{a\neq b}_{\psi\psi;f,\overline{M};g,\overline{N}}(x_p)\;;\;\hat{\mathcal{U}}^{(\alpha;\kappa)}_{\hat{F};\overline{M};\overline{N}}(x_p)\;;\;\hat{\mathfrak{v}}^{(\alpha;\kappa)}_{\hat{F};\overline{N}}(x_p)\;;\;\hat{\mathfrak{b}}^{(\hat{F})}_{(\alpha;\kappa)}(x_p)\Big)\Bigg\}\times \end{aligned} \tag{3.63}$$

$$\times\quad\exp\left\{\frac{1}{2}\;\underset{\int_C d^4x_p\;\eta_p}{\overset{a(=1,2)}{\mathrm{TR}}}\left[\underset{N_f,\hat{\gamma}^{(\mu)}_{mn},N_c}{\mathfrak{tr}}\left(\ln\left[\widetilde{\mathcal{M}}_{N;M}^{ba}(y_q,x_p)\right]-\ln\left[\begin{array}{cc}\hat{\mathcal{H}}^{11}_{N;M}(y_q,x_p) & 0\\ 0 & \hat{\mathcal{H}}^{11,T}_{N;M}(y_q,x_p)\end{array}\right]\right)\right]\right\}\times$$
$$\times\quad\exp\left\{\frac{\imath}{2}\int_C d^4x_p\;d^4y_q\;J^{\dagger,b}_{\psi;N}(y_q)\;\hat{I}\left(\left(\widetilde{\mathcal{M}}^{-\mathbf{1};ba}_{N;M}(y_q,x_p)-\left(\begin{array}{cc}\hat{\mathcal{H}}^{11;-\mathbf{1}}_{N;M}(y_q,x_p) & 0\\ 0 & \hat{\mathcal{H}}^{11,T;-\mathbf{1}}_{N;M}(y_q,x_p)\end{array}\right)\right)\right)\hat{I}\;J^{a}_{\psi;M}(x_p)\right\}\Big\rangle .$$

Corresponding to chapters 3 and 4 of Ref. [11], the coset decomposition is accomplished for $\delta\widetilde{\Sigma}_{M;N}^{ab}(x_p)$ by coset matrices $\hat{T}(x_p)$, $\hat{T}^{-1}(x_p)$ and by block diagonal self-energy densities $\delta\hat{\Sigma}^{11}_{D;M';N'}(x_p)$, $\delta\hat{\Sigma}^{22}_{D;M';N'}(x_p)$ (3.64) which are further diagonalized to the anomalous doubled eigenvalues $\delta\hat{\Lambda}(x_p)$ with block diagonal 'eigenvector' matrices $\hat{Q}(x_p)$, $\hat{Q}^{-1}(x_p)$ (3.65) (compare [11])

$$\begin{aligned}
\delta\hat{\Sigma}^{ab}_{M;N}(x_p) &= \left(\begin{array}{cc}\delta\hat{\Sigma}^{11}_{M;N}(x_p) & \imath\,\delta\hat{\Sigma}^{12}_{M;N}(x_p)\\ \imath\,\delta\hat{\Sigma}^{21}_{M;N}(x_p) & \delta\hat{\Sigma}^{22}_{M;N}(x_p)\end{array}\right) &(3.64)\\
&= \Big(\hat{T}(x_p)\Big)^{aa'}_{M;M'}\left(\begin{array}{cc}\delta\hat{\Sigma}^{11}_{D;M';N'}(x_p) & 0\\ 0 & \delta\hat{\Sigma}^{22}_{D;M';N'}(x_p)\end{array}\right)^{a'b'}_{M';N'}\Big(\hat{T}^{-1}(x_p)\Big)^{b'b}_{N';N}\\
&= \hat{T}(x_p)\;\hat{Q}^{-1}(x_p)\;\delta\hat{\Lambda}(x_p)\;\hat{Q}(x_p)\;\hat{T}^{-1}(x_p)\;;\\
\delta\hat{\Sigma}^{aa}_{D;M;N}(x_p) &= \hat{Q}^{aa;-1}_{M;M'}(x_p)\;\delta\hat{\Lambda}^{aa}_{M';N'}(x_p)\;\hat{Q}^{aa}_{N';N}(x_p)\;. &(3.65)
\end{aligned}$$

We have to require the corresponding symmetries for block diagonal self-energy densities $\delta\hat{\Sigma}^{aa}_{D;M;N}(x_p)$ following from the symmetries of densities $\delta\hat{\Sigma}^{aa}_{M;N}(x_p)$ of the original, anomalous doubled, single self-energy $\delta\widetilde{\Sigma}^{ab}_{M;N}(x_p)$ (3.44,3.45)

$$\begin{aligned}
\delta\hat{\Sigma}^{11,\dagger}_{M;N}(x_p) &= \delta\hat{\Sigma}^{11}_{M;N}(x_p)\;; & \delta\hat{\Sigma}^{22,\dagger}_{M;N}(x_p) &= \delta\hat{\Sigma}^{22}_{M;N}(x_p)\;;\\
\delta\hat{\Sigma}^{22}_{M;N}(x_p) &= -\delta\hat{\Sigma}^{11,T}_{M;N}(x_p)\;;\\
\delta\hat{\Sigma}^{11,\dagger}_{D;M;N}(x_p) &= \delta\hat{\Sigma}^{11}_{D;M;N}(x_p)\;; & \delta\hat{\Sigma}^{22,\dagger}_{D;M;N}(x_p) &= \delta\hat{\Sigma}^{22}_{D;M;N}(x_p)\;; \qquad (3.66)\\
\delta\hat{\Sigma}^{22}_{D;M;N}(x_p) &= -\delta\hat{\Sigma}^{11,T}_{D;M;N}(x_p)\;.
\end{aligned}$$

Proceeding with chapters 3, 4 of Ref. [11], the anomalous doubled eigenvalues $\delta\hat{\Lambda}^{a}(x_p)$ (3.67,3.68) of block diagonal self-energy densities $\delta\hat{\Sigma}^{aa}_{D;M;N}(x_p)$ (3.65,3.66) are given by diagonal matrices diag$\{(a=1):\,+\delta\hat{\lambda}_{N_0\times N_0}(x_p)\,\mathbf{;}\,(a=2):\,-\delta\hat{\lambda}_{N_0\times N_0}(x_p)\}$ with dimension $N_0=N_f\cdot 4_{\hat{\gamma}}\cdot(N_c=3)$, ($N_0=24,(36)$ for isospin- (flavour-) degrees of freedom). The block diagonal, diagonalizing 'eigenvector' matrices $\hat{Q}^{a=b}_{N_0\times N_0}(x_p)$ (3.69) consist of the hermitian generator $\hat{\mathcal{F}}_{D;N_0\times N_0}(x_p)$ (3.70) with vanishing diagonal because these degrees of freedom are already contained in the eigenvalues $\delta\hat{\lambda}_{N_0\times N_0}(x_p)$. Since the '22' self-energy density block $\delta\hat{\Sigma}^{22}_{D;M;N}(x_p)$, (respectively $\delta\hat{\Sigma}^{22}_{M;N}(x_p)$), is equivalent to the negative, transposed '11' self-energy density $-\delta\hat{\Sigma}^{11,T}_{D;M;N}(x_p)$ ($-\delta\hat{\Sigma}^{11,T}_{M;N}(x_p)$), we have to require symmetries (3.69) and have to construct $\hat{Q}^{22}_{N_0\times N_0}(x_p)$ by the negative, transposed generator $-\hat{\mathcal{F}}^{T}_{D;N_0\times N_0}(x_p)$ (3.71)

$$\begin{aligned}
\delta\hat{\Lambda}^{ab}(x_p) &= \delta_{ab}\;\mathrm{diag}\Big\{\underbrace{\delta\hat{\lambda}_{N_0\times N_0}(x_p)}_{a=1}\;\mathbf{;}\;\underbrace{-\delta\hat{\lambda}_{N_0\times N_0}(x_p)}_{a=2}\Big\}\;;\;N_0=N_f\cdot 4_{\hat{\gamma}}\cdot(N_c=3)\;; &(3.67)\\
\delta\hat{\lambda}_{N_0\times N_0}(x_p) &= \mathrm{diag}\Big\{\delta\hat{\lambda}_1(x_p),\;\ldots\;,\delta\hat{\lambda}_{N_0}(x_p)\Big\}\;; &(3.68)\\
\hat{Q}^{ab}_{N_0\times N_0}(x_p) &= \left(\begin{array}{cc}\hat{Q}^{11}_{N_0\times N_0}(x_p) & 0\\ 0 & \hat{Q}^{22}_{N_0\times N_0}(x_p)\end{array}\right)^{ab}\;; &(3.69)\\
\Big(\hat{Q}^{22}_{N_0\times N_0}(x_p)\Big)^{T} &= \hat{Q}^{11,\dagger}_{N_0\times N_0}(x_p)=\hat{Q}^{11,-1}_{N_0\times N_0}(x_p)\;;\\
\hat{Q}^{11}_{N_0\times N_0}(x_p) &= \exp\Big\{\imath\,\hat{\mathcal{F}}_{D;N_0\times N_0}(x_p)\Big\}\;;\quad \hat{\mathcal{F}}^{\dagger}_{D;N_0\times N_0}(x_p)=\hat{\mathcal{F}}_{D;N_0\times N_0}(x_p)\;; &(3.70)
\end{aligned}$$

$$\hat{Q}^{22}_{N_0\times N_0}(x_p) = \exp\Big\{-\imath\,\hat{\mathcal{F}}^{T}_{D;N_0\times N_0}(x_p)\Big\}\;;\quad \hat{\mathcal{F}}_{D;ii}(x_p)=0\;;(i=1,\ldots,N_0)\;. \tag{3.71}$$

In analogy to Ref. [11], the coset matrices $\hat{T}^{ab}_{M;N}(x_p)$ are specified by the generator $\hat{Y}^{ab}_{M;N}(x_p)$ with anti-symmetric sub-generators $\hat{X}_{M;N}(x_p)$, $\hat{X}^{\dagger}_{M;N}(x_p)$ of complex commuting variables for the BCS degrees of freedom

$$\hat{T}^{ab}_{M;N}(x_p) = \Big(\exp\Big\{-\hat{Y}^{a^\prime b^\prime}_{M^\prime;N^\prime}(x_p)\Big\}\Big)^{ab}_{M;N}\;; \tag{3.72}$$

$$\hat{Y}^{ab}_{M;N}(x_p) = \left(\begin{array}{cc} 0 & \hat{X}_{M;N}(x_p) \\ \hat{X}^{\dagger}_{M;N}(x_p) & 0\end{array}\right)^{a\neq b}\quad;\quad \hat{X}^{T}_{M;N}(x_p)=-\hat{X}_{M;N}(x_p)\;. \tag{3.73}$$

The generator $\hat{Y}_{2N_0\times 2N_0}(x_p)$ (3.73) with anti-symmetric sub-generators $\hat{X}_{N_0\times N_0}(x_p)$, $\hat{X}^{\dagger}_{N_0\times N_0}(x_p)$ is supplementary decomposed into block diagonal matrices $\hat{P}_{2N_0\times 2N_0}(x_p)=\hat{P}^{aa}_{N_0\times N_0}(x_p)$ (3.74) and into $\hat{Y}_{DD;2N_0\times 2N_0}(x_p)$ (3.75) with anti-symmetric, quaternion-valued, diagonal elements $\hat{X}_{DD;N_0\times N_0}(x_p)$, $\hat{\overline{f}}_{N_0\times N_0}(x_p)$ (3.76,3.77). The latter, quaternion-eigenvalues refer with the anti-symmetric Pauli matrix $(\tau_2)_{gf}$ (with the anti-symmetric parts of the $\mathrm{SU}_f(N_f=3)$ Gell-Mann matrices) to isospin- (flavour-) degrees of freedom where the complex, eigenvalue parameters $\overline{f}_{\overline{M}}(x_p)$ (3.78) are labeled by the collective index $\overline{M}=\{m,r\}$ of gamma- $\hat{\gamma}^{(\mu)}_{mn}$ and colour-matrices $\hat{t}_{\alpha;rs}$

$$\hat{Y}_{2N_0\times 2N_0}(x_p) = \hat{P}^{-1}_{2N_0\times 2N_0}(x_p)\;\hat{Y}_{DD;2N_0\times 2N_0}(x_p)\;\hat{P}_{2N_0\times 2N_0}(x_p)\;; \tag{3.74}$$

$$\hat{Y}_{DD;2N_0\times 2N_0}(x_p) = \left(\begin{array}{cc} 0 & \hat{X}_{DD;N_0\times N_0}(x_p) \\ \hat{X}^{\dagger}_{DD;N_0\times N_0}(x_p) & 0\end{array}\right)\;; \tag{3.75}$$

$$\hat{X}_{DD;N_0\times N_0}(x_p) = \hat{\overline{f}}_{N_0\times N_0}(x_p)=\hat{\overline{f}}_{g,\overline{N};f,\overline{M}}(x_p)=\big(\tau_2\big)_{gf}\,\delta_{\overline{N};\overline{M}}\,\overline{f}_{\overline{M}}(x_p)\;; \tag{3.76}$$

$$\hat{\overline{f}}_{N_0\times N_0}(x_p) = \mathrm{diag}\Big\{\big(\tau_2\big)_{gf}\,\overline{f}_1(x_p),\,\ldots\,,\big(\tau_2\big)_{gf}\,\overline{f}_{\overline{M}}(x_p),\,\ldots\,,\big(\tau_2\big)_{gf}\,\overline{f}_{N_0/2}(x_p)\Big\}\;; \tag{3.77}$$

$$\begin{aligned}\overline{f}_{\overline{M}}(x_p) &= \big|\overline{f}_{\overline{M}}(x_p)\big|\;\exp\{\imath\,\phi_{\overline{M}}(x_p)\}\;;\quad \big(\overline{f}_{\overline{M}}(x_p)\in\mathsf{C}\big)\;; \\ &\quad(\overline{M}=1,\,\ldots\,,N_0/2)\;;\;(g,f=\text{up, down, (strange)})\;; \\ \text{(inclusion of strangeness} &\;\rightarrow\; \text{with anti-symmetric SU}_f(N_f=3)\text{ matrices and }N_0/3\text{ instead of }N_0/2!)\;.\end{aligned} \tag{3.78}$$

The block diagonal eigenvector-matrices $\hat{P}^{aa}_{N_0\times N_0}(x_p)$ (3.79) of the coset generators $\hat{Y}^{ab}_{M;N}(x_p)$, $\hat{X}_{M;N}(x_p)$, $\hat{X}^{\dagger}_{M;N}(x_p)$ (3.72,3.73) have to fulfill symmetries (3.80) with a hermitian, quaternion-valued generator $\hat{\mathfrak{G}}_{D;N_0\times N_0}(x_p)$ (3.81) whose negative transposition $-\hat{\mathfrak{G}}^{T}_{D;N_0\times N_0}(x_p)$ yields the suitable generator for the '22' block $\hat{P}^{22}_{N_0\times N_0}(x_p)$ (3.82). Since $N_0/2$ complex parameters are already contained in the quaternion-valued, anti-symmetric eigenvalues $\hat{\overline{f}}_{N_0\times N_0}(x_p)$ (3.77,3.78), the analogous, quaternion-valued, diagonal matrix elements of $\hat{\mathcal{G}}_{D;f,\overline{M};g,\overline{M}}(x_p)$ have to vanish completely in the isospin- (flavour-) indices f, $g=$ up, down, (strange)

$$\hat{P}_{2N_0\times 2N_0}(x_p) = \left(\begin{array}{cc} \hat{P}^{11}_{N_0\times N_0}(x_p) & 0 \\ 0 & \hat{P}^{22}_{N_0\times N_0}(x_p)\end{array}\right)^{ab} \tag{3.79}$$

$$\Big(\hat{P}^{22}_{N_0\times N_0}(x_p)\Big)^{T} = \hat{P}^{11,\dagger}_{N_0\times N_0}(x_p)=\hat{P}^{11,-1}_{N_0\times N_0}(x_p)\;; \tag{3.80}$$

$$\hat{P}^{11}_{N_0\times N_0}(x_p) = \exp\Big\{\imath\,\hat{\mathcal{G}}_{D;N_0\times N_0}(x_p)\Big\}\;;\quad \hat{\mathcal{G}}^{\dagger}_{D;N_0\times N_0}(x_p)=\hat{\mathcal{G}}_{D;N_0\times N_0}(x_p)\;; \tag{3.81}$$

$$\hat{P}^{22}_{N_0\times N_0}(x_p) = \exp\Big\{-\imath\,\hat{\mathcal{G}}^{T}_{D;N_0\times N_0}(x_p)\Big\}\;; \tag{3.82}$$

$$\hat{\mathfrak{G}}_{D;f\overline{M};g\overline{M}}(x_p) = 0\;;\;(\overline{M}=1,\,\ldots\,,N_0/2)\;;\;(g,f=\text{up, down, (strange with }N_0/3\text{ !))}\;. \tag{3.83}$$

We apply the coset decomposition of $\delta\widetilde{\Sigma}^{ab}_{M;N}(x_p)$ in relations (3.64-3.78) to the matrix $\widetilde{\mathcal{M}}^{ba}_{N;M}(y_q,x_p)$ (3.61) on which a similarity transformation is performed by $\hat{T}(y_q)$, $\hat{T}^{-1}(x_p)$ and which comprises an effective potential $\hat{\mathcal{V}}(x_p)$ (3.60)

abbreviating the sum of the gauge fields $\hat{\mathfrak{S}}_{\alpha;\mu\nu}^{(\hat{F})}(x_p)$, $\hat{\mathfrak{B}}_{\hat{F};\beta\alpha}^{\mu\kappa}(x_p)$ and quark self-energy densities $\sigma_D^{(\alpha;\kappa)}(x_p)$

$$\begin{aligned}
\widetilde{\mathcal{M}}_{N;M}^{ba}(y_q,x_p) &= \hat{T}_{N;N'}^{bb'}(y_q)\Bigg\{\hat{T}_{N';N_1}^{-1;b'b_1}(y_q)\;\hat{I}\;\hat{S}\;\eta_q\;\frac{\hat{\mathcal{J}}_{N_1;M_1}^{b_1a_1}(y_q,x_p)}{\mathcal{N}}\;\eta_p\;\hat{S}\;\hat{I}\;\hat{T}_{M_1;M'}^{a_1a'}(x_p)+\delta^{(4)}(y_q-x_p)\;\eta_q\;\delta_{pq}\times \\
&\times\Bigg[\hat{T}_{N';N_1}^{-1;b'b_1}(x_p)\begin{pmatrix}\big[\hat{\beta}(\hat{\partial\!\!\!/}_p+\imath\hat{\mathcal{V}\!\!\!/}(x_p)-\imath\,\hat{\varepsilon}_p+\hat{m})\big]_{N_1;M_1} & 0 \\ 0 & \big[\hat{\beta}(\hat{\partial\!\!\!/}_p+\imath\hat{\mathcal{V}\!\!\!/}(x_p)-\imath\,\hat{\varepsilon}_p+\hat{m})\big]_{N_1;M_1}^T\end{pmatrix}^{b_1a_1}\hat{T}_{M_1;M'}^{a_1a'}(x_p)+ \\
&+\frac{1}{2}\begin{pmatrix}\delta\hat{\Sigma}_{D;N';M'}^{11}(x_p) & 0 \\ 0 & \delta\hat{\Sigma}_{D;N';M'}^{22}(x_p)\end{pmatrix}^{b'a'}\Bigg]\Bigg\}\hat{T}_{M';M}^{-1;a'a}(x_p)\;.
\end{aligned}\tag{3.84}$$

Taking into account the change of integration measure of the coset decomposition, we attain the generating function (3.85) with matrix $\widetilde{\mathcal{M}}_{N;M}^{ba}(y_q,x_p)$ (3.84) and with an additional polynomial $\mathfrak{P}_5(\delta\hat{\lambda}(x_p))$ of the eigenvalues $\delta\hat{\lambda}(x_p)$ (3.67,3.68) following from the Jacobian for the new integration variables

$$\begin{aligned}
Z[\hat{\mathcal{J}},J_\psi,\hat{J}_{\psi\psi},\hat{\mathfrak{j}}^{(\hat{F})}] &= \Big\langle \boldsymbol{Z}\Big[\hat{\mathcal{V}\!\!\!/}(x_p);\hat{\mathfrak{S}}^{(\hat{F})},s_\alpha,\hat{\mathfrak{B}}_{\hat{F}},\hat{\mathfrak{b}}^{(\hat{F})},\hat{\mathfrak{U}}_{\hat{F}},\hat{\mathfrak{v}}_{\hat{F}};\sigma_D;\hat{\mathfrak{j}}^{(\hat{F})};\textbf{Eq. (3.59)}\Big]\times \\
&\times\int d[\hat{T}^{-1}(x_p)\;d\hat{T}(x_p)]\int d[\delta\hat{\Sigma}_D(x_p)]\;\mathfrak{P}_5(\delta\hat{\lambda}(x_p))\times \\
&\times\Bigg\{\prod_{\{x_p\}}\boldsymbol{\Delta}\Big(\delta\widetilde{\Sigma}_{f,\overline{M};g,\overline{N}}^{ab}(x_p)+2\,\imath\hat{J}_{\psi\psi;f,\overline{M};g,\overline{N}}^{a\neq b}(x_p)\;;\;\hat{\mathfrak{U}}_{\hat{F};\overline{M};\overline{N}}^{(\alpha;\kappa)}(x_p)\;;\;\hat{\mathfrak{v}}_{\hat{F};\overline{N}}^{(\alpha;\kappa)}(x_p)\;;\;\hat{\mathfrak{b}}_{(\alpha;\kappa)}^{(\hat{F})}(x_p)\Big)\Bigg\}\times \\
&\times\exp\Bigg\{\frac{1}{2}\underset{\int_C d^4x_p\,\eta_p}{\overset{a(=1,2)}{\mathrm{TR}}}\Bigg[\underset{N_f,\hat{\gamma}_{mn}^{(\mu)},N_c}{\mathfrak{tr}}\Bigg(\ln\Big[\widetilde{\mathcal{M}}_{N;M}^{ba}(y_q,x_p)\Big]-\ln\begin{bmatrix}\hat{\mathcal{H}}_{N;M}^{11}(y_q,x_p) & 0 \\ 0 & \hat{\mathcal{H}}_{N;M}^{11,T}(y_q,x_p)\end{bmatrix}\Bigg)\Bigg]\Bigg\}\times \\
&\times\exp\Bigg\{\frac{\imath}{2}\int_C d^4x_p\;d^4y_q\;J_{\psi;N}^{\dagger,b}(y_q)\;\hat{I}\Bigg(\Bigg(\widetilde{\mathcal{M}}_{N;M}^{-1;ba}(y_q,x_p)-\begin{pmatrix}\hat{\mathcal{H}}_{N;M}^{11;-1}(y_q,x_p) & 0 \\ 0 & \hat{\mathcal{H}}_{N;M}^{11,T;-1}(y_q,x_p)\end{pmatrix}\Bigg)\Bigg)\hat{I}\;J_{\psi;M}^a(x_p)\Bigg\}\Big\rangle\,.
\end{aligned}\tag{3.85}$$

3.4 Separation of 'hinge' fields from BCS pair condensate terms

It is the aim to derive an effective Lagrangian with BCS related field degrees of freedom so that we have to remove the block diagonal self-energy densities or 'hinge' fields of the spontaneous symmetry breaking. This has to be combined with the coset decomposition SO(24 , 24) / U(24) ⊗ U(24) (for the case with 'up', 'down' isopsin degrees of freedom) of section 3.3 where we have performed a factorization (3.84) of the matrix $\widetilde{\mathcal{M}}_{N;M}^{ba}(y_q,x_p)$ into anomalous field degrees of freedom with coset matrices $\hat{T}(y_q)$, $\hat{T}^{-1}(x_p)$ and with block diagonal self-energy densities $\delta\hat{\Sigma}_{D;N;M}^{aa}(x_p)$ or 'hinge' fields. We symbolically abbreviate this factorization (3.84) by Eqs. (3.86-3.90) and introduce the gradient term $\hat{T}^{-1}\,\hat{\mathcal{H}}\,\hat{T}-\hat{\mathcal{H}}$ with anomalous doubled one-particle part $\hat{\mathcal{H}}_{N;M}^{ba}(y_q,x_p)$ (3.87,3.88) which also includes the potential part $\langle\hat{\mathcal{V}\!\!\!/}(x_p)\rangle$ (3.60)

$$\begin{aligned}
\widetilde{\mathcal{M}}_{N;M}^{ba}(y_q,x_p) &= \Big[\hat{\mathcal{H}}+\widetilde{\mathcal{J}}+\frac{1}{2}\hat{T}\;\delta\hat{\Sigma}_D\;\hat{T}^{-1}\Big]_{N;M}^{ba}(y_q,x_p)=\Big[\hat{\mathcal{H}}+\widetilde{\mathcal{J}}+\frac{1}{2}\hat{T}\;\hat{Q}^{-1}\;\delta\hat{\Lambda}\;\hat{Q}\;\hat{T}^{-1}\Big]_{N;M}^{ba}(y_q,x_p) \\
&= \hat{T}(y_q)\Bigg[\hat{\mathcal{H}}+\Big(\hat{T}^{-1}\;\hat{\mathcal{H}}\;\hat{T}-\hat{\mathcal{H}}\Big)+\widetilde{\mathcal{J}}(\hat{T}^{-1},\hat{T})+\frac{1}{2}\begin{pmatrix}\delta\hat{\Sigma}_{D;N_0\times N_0}^{11} & 0 \\ 0 & \delta\hat{\Sigma}_{D;N_0\times N_0}^{22}\end{pmatrix}\Bigg]_{N';M'}^{b'a'}(y_q,x_p)\;\;\hat{T}^{-1}(x_p)\;;
\end{aligned}\tag{3.86}$$

$$\hat{\mathcal{H}}_{N;M}^{ba}(y_q,x_p) = \delta_{pq}\;\eta_q\;\delta^{(4)}(y_q-x_p)\begin{pmatrix}\hat{H}_{N;M}(x_p) & \\ & \hat{H}_{N;M}^T(x_p)\end{pmatrix}^{ba}\;; \tag{3.87}$$

$$\hat{H}_{N;M}(x_p) = \big[\hat{\beta}\,(\hat{\partial\!\!\!/}_p+\imath\,\hat{\mathcal{V}\!\!\!/}(x_p)-\imath\,\hat{\varepsilon}_p+\hat{m}\,)\big]_{N;M}\;; \tag{3.88}$$

$$\widetilde{\mathfrak{J}}_{N;M}^{ba}(y_q,x_p) = \hat{I}\;\hat{S}\;\eta_q\;\frac{\hat{\mathfrak{J}}_{N;M}^{ba}(y_q,x_p)}{\mathcal{N}}\;\eta_p\;\hat{S}\;\hat{I}\;; \tag{3.89}$$

$$\widetilde{\mathfrak{J}}_{N;M}^{ba}\big(\hat{T}^{-1}(y_q)\,,\,\hat{T}(x_p)\big) = \Big(\hat{T}^{-1}(y_q)\;\hat{I}\;\hat{S}\;\eta_q\;\frac{\hat{\mathfrak{J}}_{N';M'}^{b'a'}(y_q,x_p)}{\mathcal{N}}\;\eta_p\;\hat{S}\;\hat{I}\;\hat{T}(x_p)\Big)_{N;M}^{ba}\,. \tag{3.90}$$

The subsequent steps (3.48-3.50), which have transformed the matrix $\widetilde{M}_{N;M}^{ba}(y_q,x_p)$ (3.47) to $\widetilde{\mathcal{M}}_{N;M}^{ba}(y_q,x_p)$ (3.51) with anti-hermitian anomalous parts, are inverted by the operations in relation (3.91)

$$\widetilde{\mathcal{M}}_{N;M}^{ba}(y_q,x_p) = \Bigg(\hat{T}(y_q)\;\hat{I}\;\hat{S}\;\underbrace{\Big(\hat{S}\;\hat{I}^{-1}\;\widetilde{\mathcal{N}}_{N';M'}^{b'a'}(y_q,x_p;\delta\hat{\Sigma}_D)\;\hat{I}^{-1}\;\hat{S}\Big)}_{\widetilde{N}_{N';M'}^{b'a'}\left(y_q,x_p;\,\hat{I}^{-1}\,\delta\hat{\Sigma}_D\,\hat{I}^{-1}\right)}\hat{S}\;\hat{I}\;\hat{T}^{-1}(x_p)\Bigg)_{N;M}^{ba}; \tag{3.91}$$

$$\widetilde{N}_{N';M'}^{b'a'}\big(y_q,x_p;\;\hat{I}^{-1}\;\delta\hat{\Sigma}_D\;\hat{I}^{-1}\big) = \Big(\hat{S}\;\hat{I}^{-1}\;\widetilde{\mathcal{N}}_{N';M'}^{b'a'}\big(y_q,x_p;\delta\hat{\Sigma}_D\big)\;\hat{I}^{-1}\;\hat{S}\Big)\,.$$

Using the factorization (3.84,3.86), one obtains a new matrix $\widetilde{N}_{N;M}^{ba}(y_q,x_p;\;\hat{I}^{-1}\;\delta\hat{\Sigma}_D\;\hat{I}^{-1})$ (3.92) under inclusion of the 'hinge' fields $\delta\hat{\Sigma}_{D;N_0\times N_0}^{11}(x_p)$, $-\delta\hat{\Sigma}_{D;N_0\times N_0}^{22}(x_p)$ (Note the minus sign before the '22' density part !)

$$\widetilde{N}_{N;M}^{ba}\big(y_q,x_p;\hat{I}^{-1}\;\delta\hat{\Sigma}_D\;\hat{I}^{-1}\big) = \widetilde{N}_{N;M}^{ba}(y_q,x_p)+\frac{1}{2}\left[\hat{S}\begin{pmatrix}\delta\hat{\Sigma}_{D;N_0\times N_0}^{11} & 0\\ 0 & -\delta\hat{\Sigma}_{D;N_0\times N_0}^{22}\end{pmatrix}\hat{S}\right]_{N;M}^{ba}(y_q,x_p)\,. \tag{3.92}$$

We separate the 'hinge' degrees of freedom $\delta\hat{\Sigma}_{D;N_0\times N_0}^{aa}(x_p)$ in $\widetilde{N}_{N;M}^{ba}(y_q,x_p;\hat{I}^{-1}\;\delta\hat{\Sigma}_D\;\hat{I}^{-1})$ (3.92) and define new matrices $\widetilde{N}_{N;M}^{ba}(y_q,x_p)$, $\hat{\mathcal{O}}_{N;M}^{ba}(y_q,x_p)$ (3.93) which only comprise BCS related field degrees of freedom in the coset matrices $\hat{T}^{-1}(y_q)$, $\hat{T}(x_p)$ with anomalous doubled one-particle and potential part $\hat{\mathcal{V}}(x_p)$ in $\hat{\mathcal{H}}_{N;M}^{ba}(y_q,x_p)$ (3.87)

$$\begin{aligned}\widetilde{N}_{N;M}^{ba}(y_q,x_p) &= \hat{S}\;\hat{I}^{-1}\;\hat{\mathcal{O}}_{N;M}^{ba}(y_q,x_p)\;\hat{I} = \hat{S}\Bigg\{\hat{\mathcal{H}}+\Big[\Big(\hat{T}(y_q)\;\hat{I}\Big)^{-1}\hat{\mathcal{H}}\Big(\hat{T}(x_p)\;\hat{I}\Big)-\hat{\mathcal{H}}\Big]+ \\ &+\;\Big(\hat{T}(y_q)\;\hat{I}\Big)^{-1}\hat{I}\;\hat{S}\;\eta_q\frac{\hat{\mathfrak{J}}_{N';M'}^{b'a'}(y_q,x_p)}{\mathcal{N}}\eta_p\;\hat{S}\;\hat{I}\Big(\hat{T}(x_p)\;\hat{I}\Big)\Bigg\}_{N;M}^{ba}(y_q,x_p)\,.\end{aligned} \tag{3.93}$$

Since we have factorized the matrix $\widetilde{\mathcal{M}}_{N;M}^{ba}(y_q,x_p)$ (3.86) in a kind of a 'similarity-transformation', the determinant (3.94) reduces to an action $\mathcal{A}'_{DET}[\hat{T},\delta\hat{\Sigma}_D,\hat{\mathcal{V}};\hat{\mathfrak{J}}]$ (3.95) where the block diagonal self-energy densities with additional minus in the '22' part appear as a summand without the coset matrices $\hat{T}^{-1}(x_p)$, $\hat{T}(x_p)$ of the BCS field degrees of freedom [10]

$$\begin{aligned}\mathrm{DET}\Big\{\widetilde{\mathcal{M}}_{N;M}^{ba}(y_q,x_p)\Big\}^{1/2} &= \mathrm{DET}\Big\{\hat{T}(y_q)\;\hat{I}\;\hat{S}\;\widetilde{N}_{N';M'}^{b'a'}(y_q,x_p;\;\hat{I}^{-1}\;\delta\hat{\Sigma}_D\;\hat{I}^{-1})\;\hat{S}\;\hat{I}\;\hat{T}^{-1}(x_p)\Big\}^{1/2} \\ &= \mathrm{DET}\Big\{\widetilde{N}_{N';M'}^{b'a'}(y_q,x_p;\;\hat{I}^{-1}\;\delta\hat{\Sigma}_D\;\hat{I}^{-1})\Big\}^{1/2} = \exp\Big\{\mathcal{A}'_{DET}\big[\hat{T},\delta\hat{\Sigma}_D,\hat{\mathcal{V}};\hat{\mathfrak{J}}\big]\Big\}\;;\end{aligned} \tag{3.94}$$

$$\mathcal{A}'_{DET}\big[\hat{T},\delta\hat{\Sigma}_D,\hat{\mathcal{V}};\hat{\mathfrak{J}}\big] = \frac{1}{2}\underset{\int_C d^4x_p\;\eta_p}{\overset{a(=1,2)}{\mathrm{TR}}}\;\underset{N_f,\hat{\gamma}_{mn}^{(\mu)},N_c}{\mathfrak{tr}}\ln\left[\widetilde{N}_{N;M}^{ba}(y_q,x_p)+\frac{1}{2}\hat{S}\begin{pmatrix}\delta\hat{\Sigma}_{D;N_0\times N_0}^{11} & 0\\ 0 & -\delta\hat{\Sigma}_{D;N_0\times N_0}^{22}\end{pmatrix}\hat{S}\right]. \tag{3.95}$$

Similarly, we apply the factorization (3.86) for the inverted matrix or Green function $\widetilde{\mathcal{M}}_{N;M}^{-1;ba}(y_q,x_p)$ and obtain relations (3.96,3.97) with the new action $\mathcal{A}'_{J_\psi}[\hat{T},\delta\hat{\Sigma}_D,\hat{\mathcal{V}};\hat{\mathfrak{J}}]$ (3.97) of the bilinear source fields $J_{\psi;N}^{\dagger,b}(y_q)$, $J_{\psi;M}^{a}(x_p)$

$$\widetilde{\mathcal{M}}_{N;M}^{-1;ba}(y_q,x_p) = \hat{T}(y_q)\;\hat{I}^{-1}\;\hat{S}\;\widetilde{N}_{N';M'}^{-1;b'a'}(y_q,x_p;\hat{I}^{-1}\;\delta\hat{\Sigma}_D\;\hat{I}^{-1})\;\hat{S}\;\hat{I}^{-1}\;\hat{T}^{-1}(x_p)\;; \tag{3.96}$$

[10] The actions $\mathcal{A}'_{DET}[\hat{T},\delta\hat{\Sigma}_D,\hat{\mathcal{V}};\hat{\mathfrak{J}}]$ (3.95), $\mathcal{A}'_{J_\psi}[\hat{T},\delta\hat{\Sigma}_D,\hat{\mathcal{V}};\hat{\mathfrak{J}}]$ (3.97) are additionally denoted by a prime '$'$' in order to point out the missing one-particle potential parts which have been moved to the background path integral (3.59) (Compare Eqs. (3.57-3.60)).

$$\mathcal{A}'_{J_\psi}[\hat{T},\delta\hat{\Sigma}_D,\hat{\mathcal{V}};\hat{\mathcal{J}}] = \frac{1}{2}\int_C d^4x_p\; d^4y_q\; J^{\dagger,b}_{\psi;N}(y_q)\;\hat{I}\;\widetilde{\mathcal{M}}^{-1;ba}_{N;M}(y_q,x_p)\;\hat{I}\;J^{a}_{\psi;M}(x_p) \tag{3.97}$$
$$= \frac{1}{2}\int_C d^4x_p\; d^4y_q\; \underbrace{J^{\dagger,b}_{\psi;N}(y_q)\big(\hat{I}\;\hat{T}(y_q)\;\hat{I}^{-1}\big)}_{\widetilde{J}^{\dagger,b'}_{\psi;N'}(y_q)}\;\hat{S}\;\widetilde{N}^{-1;b'a'}_{N';M'}(y_q,x_p;\hat{I}^{-1}\;\delta\hat{\Sigma}_D\;\hat{I}^{-1})\;\hat{S}\;\underbrace{\big(\hat{I}^{-1}\;\hat{T}^{-1}(x_p)\;\hat{I}\big)J^{a}_{\psi;M}(x_p)}_{\widetilde{J}^{a'}_{\psi;M'}(x_p)}\;.$$

The defined source field $\widetilde{J}^{a}_{\psi;M}(x_p)$ (3.98) in (3.97), which also encompasses the coset matrices, fulfills the appropriate property (3.101) for hermitian conjugation; this relation (3.101) follows from the properties (3.99,3.100) of the coset matrices under multiplication with $\hat{I}$, $\hat{I}^{-1}$

$$\widetilde{J}^{a}_{\psi;M}(x_p) = \Big(\big(\hat{I}^{-1}\;\hat{T}^{-1}(x_p)\;\hat{I}\big)J^{a'}_{\psi;M'}(x_p)\Big)^{a}_{M}\;; \tag{3.98}$$

$$\Big(\widetilde{J}^{a}_{\psi;M}(x_p)\Big)^{\dagger} = \Big(J^{\dagger,a'}_{\psi;M'}(x_p)\;\hat{I}^{-1}\;\big(\hat{T}^{-1}(x_p)\big)^{\dagger}\;\hat{I}\Big)^{a}_{M} \stackrel{?}{=} \Big(J^{\dagger,a'}_{\psi;M'}(x_p)\big(\hat{I}\;\hat{T}(x_p)\;\hat{I}^{-1}\big)\Big)^{a}_{M}\;; \tag{3.99}$$

$$\hat{I}^{-1}\;\big(\hat{T}^{-1}(x_p)\big)^{\dagger}\;\hat{I} = \hat{I}^{-1}\left[\exp\begin{pmatrix}0 & \hat{X}(x_p)\\ \hat{X}^{\dagger}(x_p) & 0\end{pmatrix}\right]^{\dagger}\hat{I} = \hat{I}^{-1}\exp\begin{pmatrix}0 & \hat{X}(x_p)\\ \hat{X}^{\dagger}(x_p) & 0\end{pmatrix}\hat{I} \tag{3.100}$$
$$= \hat{I}\exp\begin{pmatrix}0 & -\hat{X}(x_p)\\ -\hat{X}^{\dagger}(x_p) & 0\end{pmatrix}\hat{I}^{-1} = \hat{I}\;\hat{T}(x_p)\;\hat{I}^{-1}\;;$$

$$\Longrightarrow \Big(\widetilde{J}^{a}_{\psi;M}(x_p)\Big)^{\dagger} = \widetilde{J}^{\dagger,a}_{\psi;M}(x_p)\;. \tag{3.101}$$

In correspondence to Eqs. (3.94-3.101), one achieves relations (3.102,3.103) where the actions $\mathcal{A}'_{DET}\big[\hat{T},\delta\hat{\Sigma}_D,\hat{\mathcal{V}};\hat{\mathcal{J}}\big]$ (3.95), $\mathcal{A}'_{J_\psi}[\hat{T},\delta\hat{\Sigma}_D,\hat{\mathcal{V}};\hat{\mathcal{J}}]$ (3.97) are derived from the integrations with bilinear, anomalous doubled, anti-commuting Fermi fields $d[\psi^{\dagger}_{M}(x_p)\,,\,\psi_m(x_p)]$. In this manner we have inverted the various steps which have lead from the original path integral to the anomalous doubling of quark fields and to the self-energies with densities and additional BCS terms

$$Z[\hat{\mathcal{J}},J_\psi,\hat{J}_{\psi\psi},\hat{\jmath}^{(\hat{F})}] = \bigg\langle \boldsymbol{Z}\Big[\boldsymbol{\hat{\mathcal{V}}(x_p);\hat{\mathfrak{S}}^{(\hat{F})},s_\alpha,\hat{\mathfrak{B}}_{\hat{F}},\hat{\mathfrak{b}}^{(\hat{F})},\hat{\mathfrak{U}}_{\hat{F}},\hat{\mathfrak{v}}_{\hat{F}};\sigma_D;\hat{\jmath}^{(\hat{F})};}\textbf{Eq. (3.59)}\Big]\times \tag{3.102}$$
$$\times\;\int d[\hat{T}^{-1}(x_p)\;d\hat{T}(x_p)]\;\int d[\delta\hat{\Sigma}_D(x_p)]\;\;\mathfrak{P}_5(\delta\hat{\lambda}(x_p)\,)\;\times$$
$$\times\bigg\{\prod_{\{x_p\}}\boldsymbol{\Delta}\bigg(\Big(\hat{T}(x_p)\;\delta\hat{\Sigma}_D(x_p)\;\hat{T}^{-1}(x_p)\Big)^{ab}_{f,\overline{M};g,\overline{N}}+2\,\imath\hat{J}^{a\neq b}_{\psi\psi;f,\overline{M};g,\overline{N}}(x_p);\;\hat{\mathfrak{U}}^{(\alpha;\kappa)}_{\hat{F};\overline{M};\overline{N}}(x_p)\,;\;\hat{\mathfrak{v}}^{(\alpha;\kappa)}_{\hat{F};\overline{N}}(x_p)\,;\;\hat{\mathfrak{b}}^{(\hat{F})}_{(\alpha;\kappa)}(x_p)\bigg)\bigg\}\times$$
$$\times\;\;\exp\Big\{\mathcal{A}'_{DET}[\hat{T},\delta\hat{\Sigma}_D,\hat{\mathcal{V}};\hat{\mathcal{J}}]\Big\}\times\exp\Big\{\imath\,\mathcal{A}'_{J_\psi}[\hat{T},\delta\hat{\Sigma}_D,\hat{\mathcal{V}};\hat{\mathcal{J}}]\Big\}\times$$
$$\times\;\;\exp\bigg\{-\frac{1}{2}\;\underset{\int_C d^4x_p\;\eta_p}{\overset{a(=1,2)}{\mathrm{TR}}}\;\underset{N_f,\hat{\gamma}^{(\mu)}_{mn},N_c}{\mathfrak{tr}}\;\ln\Big[\hat{\mathcal{H}}^{ba}_{N;M}(y_q,x_p)\Big]\bigg\}\times$$
$$\times\;\;\exp\bigg\{-\frac{\imath}{2}\int_C d^4x_p\;d^4y_q\;J^{\dagger,b}_{\psi;N}(y_q)\;\hat{I}\;\hat{\mathcal{H}}^{-1;ba}_{N;M}(y_q,x_p)\;\hat{I}\;J^{a}_{\psi;M}(x_p)\bigg\}\bigg\rangle\;;$$

$$Z[\hat{\mathcal{J}},J_\psi,\hat{J}_{\psi\psi},\hat{\jmath}^{(\hat{F})}] = \bigg\langle \boldsymbol{Z}\Big[\boldsymbol{\hat{\mathcal{V}}(x_p);\hat{\mathfrak{S}}^{(\hat{F})},s_\alpha,\hat{\mathfrak{B}}_{\hat{F}},\hat{\mathfrak{b}}^{(\hat{F})},\hat{\mathfrak{U}}_{\hat{F}},\hat{\mathfrak{v}}_{\hat{F}};\sigma_D;\hat{\jmath}^{(\hat{F})};}\textbf{Eq. (3.59)}\Big]\times \tag{3.103}$$
$$\times\;\int d[\hat{T}^{-1}(x_p)\;d\hat{T}(x_p)]\;\int d[\delta\hat{\Sigma}_D(x_p)]\;\;\mathfrak{P}_5(\delta\hat{\lambda}(x_p)\,)\;\times\int d[\psi^{\dagger}_{M}(x_p),\;\psi_M(x_p)]\;\times$$

$$\times\Big\{\prod_{\{x_p\}}\boldsymbol{\Delta}\Big(\Big(\hat{T}(x_p)\;\delta\hat{\Sigma}_D(x_p)\;\hat{T}^{-1}(x_p)\Big)^{ab}_{f,\overline{M};g,\overline{N}}+2\,\imath\hat{J}^{a\neq b}_{\psi\psi;f,\overline{M};g,\overline{N}}(x_p)\;;\;\hat{\mathfrak{U}}^{(\alpha;\kappa)}_{\hat{F};\overline{M};\overline{N}}(x_p)\;;\;\hat{\mathfrak{v}}^{(\alpha;\kappa)}_{\hat{F};\overline{N}}(x_p)\;;\;\hat{\mathfrak{b}}^{(\hat{F})}_{(\alpha;\kappa)}(x_p)\Big)\Big\}\times$$
$$\times\;\;\exp\Big\{-\frac{\imath}{2}\int_C d^4x_p\;d^4y_q\;\Psi^{\dagger,b}_N(y_q)\Big[\widetilde{N}^{ba}_{N;M}(y_q,x_p)+\frac{1}{2}\;\hat{S}\begin{pmatrix}\delta\hat{\Sigma}^{11}_{D;N_0\times N_0} & 0\\ 0 & -\delta\hat{\Sigma}^{22}_{D;N_0\times N_0}\end{pmatrix}\hat{S}\Big]\Psi^a_M(x_p)\Big\}\times$$
$$\times\;\;\exp\Big\{-\frac{\imath}{2}\int_C d^4x_p\;d^4y_q\;\widetilde{J}^{\dagger,a}_{\psi;M}(x_p)\;\hat{S}\;\Psi^a_M(x_p)+\Psi^{\dagger,a}_M(x_p)\;\hat{S}\;\widetilde{J}^a_{\psi;M}(x_p)\Big\}\times$$
$$\times\;\;\exp\Big\{-\frac{1}{2}\;\underset{\int_C d^4x_p\;\eta_p}{\overset{a(=1,2)}{\mathrm{TR}}}\;\underset{N_f,\hat{\gamma}^{(\mu)}_{mn},N_c}{\mathfrak{tr}}\ln\Big[\hat{\mathcal{H}}^{ba}_{N;M}(y_q,x_p)\Big]\Big\}\times$$
$$\times\;\;\exp\Big\{-\frac{\imath}{2}\int_C d^4x_p\;d^4y_q\;J^{\dagger,b}_{\psi;N}(y_q)\;\hat{I}\;\hat{\mathcal{H}}^{-\mathbf{1};ba}_{N;M}(y_q,x_p)\;\hat{I}\;J^a_{\psi;M}(x_p)\Big\}\Big\rangle\;;$$
$$\widetilde{J}^a_{\psi;M}(x_p)=\Big(\hat{I}^{-1}\;\hat{T}^{-1}(x_p)\;\hat{I}\;J^{a'}_{\psi;M'}(x_p)\Big)^a_M\;;\qquad\widetilde{J}^{\dagger,a}_{\psi;M}(x_p)=\Big(J^{\dagger,a'}_{\psi;M'}(x_p)\;\hat{I}\;\hat{T}(x_p)\;\hat{I}^{-1}\Big)^a_M\;.\qquad(3.104)$$

We note that the part (3.105) with the 'hinge' fields $\delta\hat{\Sigma}^{11}_{D;N;M}(x_p)$, $-\delta\hat{\Sigma}^{22}_{D;N;M}(x_p)$ and anti-commuting, anomalous doubled Fermi fields does not contribute in the generating functions (3.102,3.103); therefore, we have accomplished a *projection* onto the BCS related field degrees of freedom with the coset matrices $\hat{T}^{-1}(x_p)$, $\hat{T}(x_p)$; it has to be emphasized that this operation is not invertible according to standard properties of projections which do not allow the construction of any inverses

$$\exp\Big\{-\frac{\imath}{2}\int_C d^4x_p\;\underbrace{\Psi^{\dagger,b}_N(x_p)\;\frac{1}{2}\;\hat{S}\begin{pmatrix}\delta\hat{\Sigma}^{11}_{D;N;M}(x_p) & 0\\ 0 & -\delta\hat{\Sigma}^{22}_{D;N;M}(x_p)\end{pmatrix}\hat{S}\;\Psi^a_M(x_p)}_{\equiv 0}\Big\}\equiv 1\;.\qquad(3.105)$$

After removal of part (3.105) from (3.103), we can again perform integrations of anomalous doubled Grassmann fields and attain the path integral (3.106) with sub-generating function $Z_{\hat{J}_{\psi\psi}}[\hat{T}]$ (3.107) of the BCS related source field $\hat{J}^{b\neq a}_{\psi\psi;N;M}(x_p)$. The matrix $\widetilde{N}^{ba}_{N;M}(y_q,x_p)$ remains in the actions $\mathcal{A}_{DET}[\hat{T},\hat{\mathcal{V}};\hat{\mathcal{J}}]$, $\mathcal{A}_{J_\psi}[\hat{T},\hat{\mathcal{V}};\hat{\mathcal{J}}]$, but without any 'hinge' or block diagonal self-energy density degrees of freedom

$$Z[\hat{\mathcal{J}},J_\psi,\hat{J}_{\psi\psi},\hat{\mathfrak{j}}^{(\hat{F})}]\;=\;\Big\langle\boldsymbol{Z}\Big[\boldsymbol{\hat{\mathcal{V}}(x_p);\hat{\mathfrak{S}}^{(\hat{F})},s_\alpha,\hat{\mathfrak{B}}_{\hat{F}},\hat{\mathfrak{b}}^{(\hat{F})},\hat{\mathfrak{U}}_{\hat{F}},\hat{\mathfrak{v}}_{\hat{F}};\sigma_D;\hat{\mathfrak{j}}^{(\hat{F})};}\textbf{Eq. (3.59)}\Big]\times\qquad(3.106)$$
$$\times\;\int d[\hat{T}^{-1}(x_p)\;d\hat{T}(x_p)]\;\;Z_{\hat{J}_{\psi\psi}}[\hat{T}]\;\exp\Big\{\mathcal{A}_{DET}[\hat{T},\hat{\mathcal{V}};\hat{\mathcal{J}}]\Big\}\;\exp\Big\{\imath\,\mathcal{A}_{J_\psi}[\hat{T},\hat{\mathcal{V}};\hat{\mathcal{J}}]\Big\}\Big\rangle\;;$$
$$Z_{\hat{J}_{\psi\psi}}[\hat{T}]\;=\;\int d[\delta\hat{\Sigma}_D(x_p)]\;\mathfrak{P}_5\big(\delta\hat{\lambda}(x_p)\big)\;\times\qquad(3.107)$$
$$\times\Big\{\prod_{\{x_p\}}\boldsymbol{\Delta}\Big(\Big(\hat{T}(x_p)\;\delta\hat{\Sigma}_D(x_p)\;\hat{T}^{-1}(x_p)\Big)^{ab}_{f,\overline{M};g,\overline{N}}+2\,\imath\hat{J}^{a\neq b}_{\psi\psi;f,\overline{M};g,\overline{N}}(x_p)\;;\;\hat{\mathfrak{U}}^{(\alpha;\kappa)}_{\hat{F};\overline{M};\overline{N}}(x_p)\;;\;\hat{\mathfrak{v}}^{(\alpha;\kappa)}_{\hat{F};\overline{N}}(x_p)\;;\;\hat{\mathfrak{b}}^{(\hat{F})}_{(\alpha;\kappa)}(x_p)\Big)\Big\};$$
$$\mathcal{A}_{DET}[\hat{T},\hat{\mathcal{V}};\hat{\mathcal{J}}]\;=\;\frac{1}{2}\;\underset{\int_C d^4x_p\;\eta_p}{\overset{a(=1,2)}{\mathrm{TR}}}\;\underset{N_f,\hat{\gamma}^{(\mu)}_{mn},N_c}{\mathfrak{tr}}\Big(\ln\Big[\widetilde{N}^{ba}_{N;M}(y_q,x_p)\Big]-\ln\Big[\hat{\mathcal{H}}^{ba}_{N;M}(y_q,x_p)\Big]\Big)\;;\qquad(3.108)$$
$$\mathcal{A}_{J_\psi}[\hat{T},\hat{\mathcal{V}};\hat{\mathcal{J}}]\;=\;\frac{1}{2}\int_C d^4x_p\;d^4y_q\;\times\qquad(3.109)$$
$$\times\;J^{\dagger,b}_{\psi;N}(y_q)\;\hat{I}\Big(\hat{T}(y_q)\;\hat{I}^{-1}\;\hat{S}\;\widetilde{N}^{-\mathbf{1};b'a'}_{N';M'}(y_q,x_p)\;\hat{S}\;\hat{I}^{-1}\;\hat{T}^{-1}(x_p)-\hat{\mathcal{H}}^{-\mathbf{1};ba}_{N;M}(y_q,x_p)\Big)\;\hat{I}\;J^a_{\psi;M}(x_p)\;.$$

The matrix $\widetilde{N}_{N;M}^{ba}(y_q,x_p)$ is further simplified to the related matrix $\hat{\mathfrak{O}}_{N;M}^{ba}(y_q,x_p)$ which has the equivalent determinant (3.112) and similar time contour Green function (3.113). We define the actions $\mathcal{A}_{DET}[\hat{T},\hat{\mathcal{V}};\hat{\mathcal{J}}]$ (3.114), $\mathcal{A}_{J_\psi}[\hat{T},\hat{\mathcal{V}};\hat{\mathcal{J}}]$ (3.115) in terms of $\hat{\mathfrak{O}}_{N;M}^{ba}(y_q,x_p)$ (3.111) with gradient part $\hat{T}^{-1}\,\hat{\mathfrak{H}}\,\hat{T}-\hat{\mathfrak{H}}$ and can finally cease with the effective path integral (3.116) which contains the coset matrices for the BCS terms as the only remaining field degrees of freedom. The gauge field and quark self-energy density $\sigma_D^{(\alpha;\kappa)}(x_p)$ degrees of freedom are incorporated in the averaging with the background functional (3.59) and the sub-generating function $Z_{\hat{J}_{\psi\psi}}[\hat{T}]$ (3.107)

$$
\begin{aligned}
\widetilde{N}_{N;M}^{ba}(y_q,x_p) &= \hat{S}\;\hat{I}^{-1}\;\hat{\mathfrak{O}}_{N;M}^{ba}(y_q,x_p)\;\hat{I}\;; && (3.110)\\
\hat{\mathfrak{O}}_{N;M}^{ba}(y_q,x_p) &=\bigg\{\hat{\mathfrak{H}}+\Big(\hat{T}^{-1}\hat{\mathfrak{H}}\hat{T}-\hat{\mathfrak{H}}\Big)+\hat{T}^{-1}\;\hat{I}\;\hat{S}\;\eta_q\frac{\hat{\mathcal{J}}_{N';M'}^{b'a'}(y_q,x_p)}{\mathcal{N}}\eta_p\;\hat{S}\;\hat{I}\;\hat{T}\bigg\}_{N;M}^{ba}(y_q,x_p)\;; && (3.111)\\
\mathrm{DET}\Big[\widetilde{N}_{N;M}^{ba}(y_q,x_p)\Big] &= \mathrm{DET}\Big[\hat{\mathfrak{O}}_{N;M}^{ba}(y_q,x_p)\Big]\;; && (3.112)\\
\widetilde{N}_{N;M}^{-\mathbf{1};ba}(y_q,x_p) &= \hat{I}^{-1}\;\hat{\mathfrak{O}}_{N;M}^{-\mathbf{1};ba}(y_q,x_p)\;\hat{I}\;\hat{S}\;; && (3.113)\\
\mathcal{A}_{DET}[\hat{T},\hat{\mathcal{V}};\hat{\mathcal{J}}] &= \frac{1}{2}\underset{\int_C d^4x_p\;\eta_p}{\overset{a(=1,2)}{\mathrm{TR}}}\;\underset{N_f,\hat{\gamma}_{mn}^{(\mu)},N_c}{\mathfrak{tr}}\bigg(\ln\Big[\hat{\mathfrak{O}}_{N;M}^{ba}(y_q,x_p)\Big]-\ln\Big[\hat{\mathfrak{H}}_{N;M}^{ba}(y_q,x_p)\Big]\bigg)\;; && (3.114)\\
\mathcal{A}_{J_\psi}[\hat{T},\hat{\mathcal{V}};\hat{\mathcal{J}}] &= \frac{1}{2}\int_C d^4x_p\;d^4y_q\;\times && (3.115)\\
&\times J_{\psi;N}^{\dagger,b}(y_q)\;\hat{I}\bigg(\hat{T}(y_q)\;\hat{\mathfrak{O}}_{N';M'}^{-\mathbf{1};b'a'}(y_q,x_p)\;\hat{T}^{-1}(x_p)-\hat{\mathfrak{H}}_{N;M}^{-\mathbf{1};ba}(y_q,x_p)\bigg)\;\hat{I}\;J_{\psi;M}^{a}(x_p)\;;\\
Z[\hat{\mathcal{J}},J_\psi,\hat{J}_{\psi\psi},\hat{\jmath}^{(\hat{F})}] &= \bigg\langle\boldsymbol{Z}\Big[\hat{\boldsymbol{\mathcal{V}}}(\boldsymbol{x_p});\hat{\boldsymbol{\mathfrak{S}}}^{(\hat{\boldsymbol{F}})},\boldsymbol{s_\alpha},\hat{\boldsymbol{\mathfrak{B}}}_{\hat{\boldsymbol{F}}},\hat{\boldsymbol{\mathfrak{b}}}^{(\hat{\boldsymbol{F}})},\hat{\boldsymbol{\mathfrak{U}}}_{\hat{\boldsymbol{F}}},\hat{\boldsymbol{\mathfrak{v}}}_{\hat{\boldsymbol{F}}};\boldsymbol{\sigma_D};\hat{\boldsymbol{\jmath}}^{(\hat{\boldsymbol{F}})};\textbf{Eq. (3.59)}\Big]\times && (3.116)\\
&\times\int d[\hat{T}^{-1}(x_p)\;d\hat{T}(x_p)]\;\;Z_{\hat{J}_{\psi\psi}}[\hat{T}]\;\exp\Big\{\mathcal{A}_{DET}[\hat{T},\hat{\mathcal{V}};\hat{\mathcal{J}}]\Big\}\;\exp\Big\{\imath\,\mathcal{A}_{J_\psi}[\hat{T},\hat{\mathcal{V}};\hat{\mathcal{J}}]\Big\}\bigg\rangle\;.
\end{aligned}
$$

4 Infinite order gradient expansion to an effective action

4.1 Separation into path integrals of BCS terms with coset matrices and density related parts

Although we have performed several involved HST's to self-energies and a coset decomposition in section 3, the finally obtained, exact relations (3.110-3.116) are remarkable because they indicate a clear separation of the original path integral (2.15-2.32) into a density related part with generating function (3.59) and into BCS degrees of freedom with coset matrices $\hat{T}(x_p)=\exp\{-\hat{Y}(x_p)\}$ (3.72-3.83). The composed gauge field $\mathcal{V}_{\alpha;\mu}(x_p)$ (3.60) appears in both parts of the total path integral (3.116) and replaces the original gauge fields $A_{\alpha;\mu}(x_p)$ (2.1-2.6) with auxiliary real field $s_\alpha(x_p)$ for axial gauge fixing. One can even prove a gauge invariance between $\mathcal{V}_{\alpha;\mu}(x_p)$ and the coset matrices $\hat{T}(x_p)=\exp\{-\hat{Y}(x_p)\}$ of (3.116) in a classical consideration where a chosen gauge condition for the composed field $\mathcal{V}_{\alpha;\mu}(x_p)$ is achieved by adaption of the auxiliary real field $s_\gamma(x_p)$ with a shift of its value (compare Eq. (3.60)). In the quantum mechanical case, we obtain a Ward identity of the derived path integral (3.116) with background field averaging (3.59) and with projection $\hat{S}$ onto BCS terms in the actions $\mathcal{A}_{DET}[\hat{T},\hat{\mathcal{V}};\hat{\mathcal{J}}]$, $\mathcal{A}_{J_\psi}[\hat{T},\hat{\mathcal{V}};\hat{\mathcal{J}}]$ (cf appendix B). Although there appears no action of a field strength tensor as $\hat{F}_\alpha^{\mu\nu}(x_p)$ (2.5) for the composed gauge field $\mathcal{V}_\alpha^\mu(x_p)$, a gauge invariance follows because the change of actions with $\delta\mathcal{V}_\alpha^\mu(x_p)$ in a gauge transformation is compensated by the change of the coset matrices $\hat{T}(x_p)=\exp\{-\hat{Y}(x_p)\}$. In this respect the actions $\mathcal{A}_{DET}[\hat{T},\hat{\mathcal{V}};\hat{\mathcal{J}}]$, $\mathcal{A}_{J_\psi}[\hat{T},\hat{\mathcal{V}};\hat{\mathcal{J}}]$ of the coset matrices replace the action of a quadratic field strength tensor for the composed gauge field.

According to the separation into density and BCS terms, we split the total path integral $Z[\hat{\mathcal{J}},J_\psi,\hat{J}_{\psi\psi},\hat{\jmath}^{(\hat{F})}]$ (3.116) into the density related generating function (3.59) which allows to determine a mean field solution $\langle\mathcal{V}_{\alpha;\mu}(x_p)\rangle_{\text{eq. (3.59)}}$;

this assumed particular solution $\langle\mathcal{V}_{\alpha;\mu}(x_p)\,\rangle_{\text{eq. }(3.59)}$ comprises real and imaginary parts where the sign of the imaginary values of $\langle\mathcal{V}_{\alpha;\mu}(x_p)\,\rangle_{(3.59)}$ has to comply with the anti-hermitian '$-\imath\,\hat{\varepsilon}_p$' terms (2.27) for stable propagation of the coset matrices $\hat{T}(x_p)=\exp\{-\hat{Y}(x_p)\,\}$. This definite, fixed mean field solution $\langle\mathcal{V}_{\alpha;\mu}(x_p)\,\rangle_{(3.59)}$ is inserted into the actions $\mathcal{A}_{DET}[\hat{T},\langle\hat{\mathcal{V}}\rangle_{(3.59)};\hat{\mathcal{J}}]$, $\mathcal{A}_{J_\psi}[\hat{T},\langle\hat{\mathcal{V}}\rangle_{(3.59)};\hat{\mathcal{J}}]$, where it enters into the one-particle operators $\hat{H}(x_p)$, $\hat{H}^T(x_p)$ or more precisely into the anomalous doubled version $\hat{\mathcal{H}}(x_p)$ (3.87,3.88)

$$\begin{aligned}
Z[\hat{\mathcal{J}},J_\psi,\hat{J}_{\psi\psi},\hat{\jmath}^{(\hat{F})}] &= \Bigg\langle \boldsymbol{Z\Big[\hat{\mathcal{V}}(x_p);\hat{\mathfrak{S}}^{(\hat{F})},s_\alpha,\hat{\mathfrak{B}}_{\hat{F}},\hat{\mathfrak{b}}^{(\hat{F})},\hat{\mathfrak{U}}_{\hat{F}},\hat{\mathfrak{v}}_{\hat{F}};\sigma_D;\hat{\jmath}^{(\hat{F})};\textbf{Eq. (3.59)}\Big]}\times \qquad (4.1)\\
&\times\int d[\hat{T}^{-1}(x_p)\,d\hat{T}(x_p)]\;\;Z_{\hat{J}_{\psi\psi}}[\hat{T}]\;\exp\Big\{\mathcal{A}_{DET}[\hat{T},\hat{\mathcal{V}};\hat{\mathcal{J}}]\Big\}\;\exp\Big\{\imath\,\mathcal{A}_{J_\psi}[\hat{T},\hat{\mathcal{V}};\hat{\mathcal{J}}]\Big\}\Bigg\rangle\\
&\approx\underbrace{Z\Big[\hat{\mathcal{V}}(x_p);\hat{\mathfrak{S}}^{(\hat{F})},s_\alpha,\hat{\mathfrak{B}}_{\hat{F}},\hat{\mathfrak{b}}^{(\hat{F})},\hat{\mathfrak{U}}_{\hat{F}},\hat{\mathfrak{v}}_{\hat{F}};\sigma_D;\hat{\jmath}^{(\hat{F})};\textbf{Eq. (3.59)}\Big]}_{\Longrightarrow\text{classical solution }\langle\mathcal{V}_{\alpha;\mu}(x_p)\,\rangle_{(3.59)}}\times\\
&\times\int d[\hat{T}^{-1}(x_p)\,d\hat{T}(x_p)]\;\;\Big\langle Z_{\hat{J}_{\psi\psi}}[\hat{T}]\Big\rangle_{(3.59)}\;\exp\Big\{\mathcal{A}_{DET}[\hat{T},\langle\hat{\mathcal{V}}\rangle_{(3.59)};\hat{\mathcal{J}}]\Big\}\;\exp\Big\{\imath\,\mathcal{A}_{J_\psi}[\hat{T},\langle\hat{\mathcal{V}}\rangle_{(3.59)};\hat{\mathcal{J}}]\Big\}\;;
\end{aligned}$$

$$\mathcal{A}_{DET}[\hat{T},\langle\hat{\mathcal{V}}\rangle_{(3.59)};\hat{\mathcal{J}}] = \frac{1}{2}\;\underset{\int_C d^4x_p\,\eta_p}{\overset{a(=1,2)}{\mathrm{TR}}}\;\underset{N_f,\hat{\gamma}_{mn}^{(\mu)},N_c}{\mathfrak{tr}}\Bigg(\ln\Big[\Big\langle\hat{\mathcal{O}}_{N;M}^{ba}(y_q,x_p)\Big\rangle_{(3.59)}\Big]-\ln\Big[\Big\langle\hat{\mathcal{H}}_{N;M}^{ba}(y_q,x_p)\Big\rangle_{(3.59)}\Big]\Bigg); \qquad (4.2)$$

$$\begin{aligned}
\mathcal{A}_{J_\psi}[\hat{T},\langle\hat{\mathcal{V}}\rangle_{(3.59)};\hat{\mathcal{J}}] &= \frac{1}{2}\int_C d^4x_p\;d^4y_q\;\times \qquad (4.3)\\
&\times J_{\psi;N}^{\dagger,b}(y_q)\;\hat{I}\Bigg(\hat{T}(y_q)\;\Big\langle\hat{\mathcal{O}}_{N';M'}^{b'a'}(y_q,x_p)\Big\rangle_{(3.59)}^{\boldsymbol{-1}}\;\hat{T}^{-1}(x_p)-\Big\langle\hat{\mathcal{H}}_{N;M}^{ba}(y_q,x_p)\Big\rangle_{(3.59)}^{\boldsymbol{-1}}\Bigg)\hat{I}\;J_{\psi;M}^{a}(x_p)\;;
\end{aligned}$$

$$\begin{aligned}
&\Big\langle\hat{\mathcal{O}}_{N;M}^{ba}(y_q,x_p)\Big\rangle_{(3.59)} = \qquad (4.4)\\
&=\;\;\Bigg\{\langle\hat{\mathcal{H}}\rangle_{(3.59)}+\underbrace{\Big(\hat{T}^{-1}\langle\hat{\mathcal{H}}\rangle_{(3.59)}\hat{T}-\langle\hat{\mathcal{H}}\rangle_{(3.59)}\Big)}_{\Delta\langle\hat{\mathcal{H}}\rangle_{(3.59)}}+\hat{T}^{-1}\;\hat{I}\;\hat{S}\;\eta_q\frac{\hat{\mathcal{J}}_{N';M'}^{b'a'}(y_q,x_p)}{\mathcal{N}}\eta_p\;\hat{S}\;\hat{I}\;\hat{T}\Bigg\}_{N;M}^{ba}(y_q,x_p)\;;
\end{aligned}$$

$$\Delta\langle\hat{\mathcal{H}}\rangle_{(3.59)} = \hat{T}^{-1}\;\langle\hat{\mathcal{H}}\rangle_{(3.59)}\;\hat{T}-\langle\hat{\mathcal{H}}\rangle_{(3.59)}\;. \qquad (4.5)$$

The background gauge field $\mathcal{V}_\alpha^\mu(x_p)$ consists of the self-energy field strength tensor $\hat{\mathfrak{S}}_{\alpha;\mu\nu}^{(\hat{F})}(x_p)$, its derivative and its inverse with structure constants $C_{\alpha\beta\gamma}$ and additionally of the self-energy quark densities $\sigma_D^{(\alpha;\kappa)}(x_p)$ which are dressed by the eigenvectors $\hat{\mathfrak{B}}_{\hat{F};\beta(\alpha)}^{\mu(\kappa)}(x_p)$ of the self-energy field strength term $C_{\alpha\beta\gamma}\;\hat{\mathfrak{S}}_{\alpha;\mu\nu}^{(\hat{F})}(x_p)$

$$\begin{aligned}
\mathcal{V}_\beta^\mu(x_p) &= \Big[\Big(\hat{\partial}_p^\lambda\hat{\mathfrak{S}}_{\gamma;\nu\lambda}^{(\hat{F})}(x_p)\Big)-s_\gamma(x_p)\;n_\nu\Big]\;\Big[-\imath\,\hat{\mathfrak{e}}_p^{(\hat{F})}+C_{\alpha\beta'\gamma'}\;\hat{\mathfrak{S}}_\alpha^{(\hat{F})\mu'\nu'}(x_p)\Big]_{\gamma\beta}^{\boldsymbol{-1};\nu\mu}+ \qquad (4.6)\\
&+\;\;\frac{1}{2}\sum_{(\alpha)=1,..,8}^{(\kappa)=0,..,3}\hat{\mathfrak{B}}_{\hat{F};\beta(\alpha)}^{\mu(\kappa)}(x_p)\;\sigma_D^{(\alpha;\kappa)}(x_p)\;;\\
C_{\alpha\beta\gamma}\;\hat{\mathfrak{S}}_\alpha^{(\hat{F})\mu\nu}(x_p) &:= \sum_{(\alpha)=1,..,8}^{(\kappa)=0,..,3}\hat{\mathfrak{B}}_{\hat{F};\beta(\alpha)}^{\mu(\kappa)}(x_p)\;\hat{\mathfrak{b}}_{(\alpha;\kappa)}^{(\hat{F})}(x_p)\;\hat{\mathfrak{B}}_{\hat{F};(\alpha)\gamma}^{T,(\kappa)\nu}(x_p)\;; \qquad (4.7)\\
d[\hat{\mathfrak{S}}_\alpha^{(\hat{F})\mu\nu}(x_p)] &\to d[\hat{\mathfrak{S}}_\alpha^{(\hat{F})\mu\nu}(x_p);\hat{\mathfrak{B}}_{\hat{F};\beta(\alpha)}^{\mu(\kappa)}(x_p);\hat{\mathfrak{b}}_{(\alpha;\kappa)}^{(\hat{F})}(x_p)]\;; \qquad (4.8)
\end{aligned}$$

$$d[\hat{\mathfrak{S}}_{\alpha}^{(\hat{F})\mu\nu}(x_p);\hat{\mathfrak{B}}_{\hat{F};\beta(\alpha)}^{\mu(\kappa)}(x_p);\hat{\mathfrak{b}}_{(\alpha;\kappa)}^{(\hat{F})}(x_p)]=d[\hat{\mathfrak{S}}_{\alpha}^{(\hat{F})\mu\nu}(x_p)]\;\;d[\hat{\mathfrak{B}}_{\hat{F};\beta(\alpha)}^{\mu(\kappa)}(x_p);\hat{\mathfrak{b}}_{(\alpha;\kappa)}^{(\hat{F})}(x_p)]\times \tag{4.9}$$
$$\times\quad\Big\{\prod_{\{x_p\}}\delta\Big(C_{\alpha\beta\gamma}\;\hat{\mathfrak{S}}_{\alpha}^{(\hat{F})\mu\nu}(x_p)-\sum_{(\alpha)=1,..,8}^{(\kappa)=0,..,3}\hat{\mathfrak{B}}_{\hat{F};\beta(\alpha)}^{\mu(\kappa)}(x_p)\;\hat{\mathfrak{b}}_{(\alpha;\kappa)}^{(\hat{F})}(x_p)\;\hat{\mathfrak{B}}_{\hat{F};(\alpha)\gamma}^{T,(\kappa)\nu}(x_p)\Big)\Big\}\,.$$

The anomalous doubled one-particle operator $\hat{\mathcal{H}}(x_p)$ contains the composed gauge fields $\mathcal{V}_{\alpha}^{\mu}(x_p)$, $\hat{\not{\mathcal{V}}}(x_p)$ or more precisely the anomalous doubled version $\hat{\mathcal{V}}^{\mu}(x_p)=\hat{\mathfrak{T}}_{\alpha}\;\mathcal{V}_{\alpha}^{\mu}(x_p)$, $\hat{\not{\mathcal{V}}}(x_p)$ (4.14) with extended generators $\hat{\mathfrak{T}}_{\alpha}$ of $\mathrm{SU}_c(N_c=3)$ which are doubled by the transpose $\hat{t}_{\alpha}^{T}$ in the '22' block. In analogy we double the Dirac gamma matrices $\hat{\beta}$, $\hat{\gamma}^{\mu}$ to their extended block diagonal forms $\hat{\mathcal{B}}^{aa}$, $\hat{\Gamma}^{\mu,aa}$ (4.13) according to the anomalous doubling of Fermi fields

$$\begin{aligned}
\hat{\mathcal{H}}_{N;M}^{ba}(y_q,x_p) &= \delta^{(4)}(y_q-x_p)\;\eta_q\;\delta_{qp}\begin{pmatrix}\hat{H}_{N;M}(x_p) & \\ & \hat{H}_{N;M}^{T}(x_p)\end{pmatrix}^{ba}; && (4.10)\\
\hat{H}(x_p) &= \Big[\hat{\beta}\Big(\hat{\not{\partial}}_p+\imath\,\hat{\not{\mathcal{V}}}(x_p)-\imath\,\hat{\varepsilon}_p+\hat{m}\Big)\Big]\;;\;(\hat{\varepsilon}_p=\hat{\beta}\;\varepsilon_p=\hat{\beta}\;\eta_p\;\varepsilon_+\;;\;\varepsilon_+>0)\;; && (4.11)\\
&= \hat{\beta}\hat{\gamma}^{\mu}\;\hat{\partial}_{p,\mu}+\imath\,\hat{\beta}\hat{\gamma}^{\mu}\;\hat{t}_{\alpha}\;\mathcal{V}_{\alpha}^{\mu}(x_p)+\hat{\beta}\;\hat{m}-\imath\;\varepsilon_p\;\hat{1}_{N_0\times N_0}\;;\\
\hat{H}^{T}(x_p) &= \Big[\hat{\beta}\Big(\hat{\not{\partial}}_p+\imath\,\hat{\not{\mathcal{V}}}(x_p)-\imath\,\hat{\varepsilon}_p+\hat{m}\Big)\Big]^{T} && (4.12)\\
&= -\big(\hat{\beta}\hat{\gamma}^{\mu}\big)^{T}\;\hat{\partial}_{p,\mu}-\imath\,\big(\hat{\beta}\hat{\gamma}^{\mu}\big)^{T}\;(-\hat{t}_{\alpha}^{T})\;\mathcal{V}_{\alpha}^{\mu}(x_p)-\big(\hat{\beta}\,(-\hat{m})\big)^{T}-\imath\;\varepsilon_p\;\hat{1}_{N_0\times N_0}\;;\\
\hat{\mathcal{B}}\;\hat{\Gamma}^{\mu} &= \begin{pmatrix}\hat{\beta}\;\hat{\gamma}^{\mu} & \\ & (\,\hat{\beta}\;\hat{\gamma}^{\mu}\,)^{T}\end{pmatrix}^{ba};\;\hat{\mathcal{B}}=\begin{pmatrix}\hat{\beta} & \\ & \hat{\beta}^{T}\end{pmatrix}^{ba};\;\hat{\mathcal{B}}\hat{M}=\begin{pmatrix}(\hat{\beta}\,\hat{m}) & \\ & (\hat{\beta}\,(-\hat{m})\,)^{T}\end{pmatrix}^{ba}; && (4.13)\\
\hat{\not{\mathcal{V}}}(x_p) &= \hat{\Gamma}^{\mu}\;\hat{\mathcal{V}}_{\mu}(x_p)=\hat{\Gamma}^{\mu}\;\hat{\mathfrak{T}}_{\alpha}\;\mathcal{V}_{\alpha;\mu}(x_p)\;;\quad\hat{\Gamma}^{0}=\begin{pmatrix}\hat{\gamma}^{0} & 0\\ 0 & \hat{\gamma}^{0,T}\end{pmatrix};\quad\hat{\vec{\Gamma}}=\begin{pmatrix}\hat{\vec{\gamma}} & 0\\ 0 & -\hat{\vec{\gamma}}^{T}\end{pmatrix}; && (4.14)\\
\hat{\mathfrak{T}}_{\alpha} &= \begin{pmatrix}\hat{t}_{\alpha} & \\ & -\hat{t}_{\alpha}^{T}\end{pmatrix}^{ba};\hat{\mathcal{V}}^{\mu}(x_p)=\hat{\mathfrak{T}}_{\alpha}\;\mathcal{V}_{\alpha}^{\mu}(x_p)\;;\\
\hat{\mathcal{H}}(x_p) &= \hat{S}\Big(\hat{\mathcal{B}}\;\hat{\Gamma}^{\mu}\;\hat{\partial}_{p,\mu}+\hat{\mathcal{B}}\;\big(\imath\,\hat{\not{\mathcal{V}}}(x_p)+\hat{M}\big)\Big)-\imath\;\varepsilon_p\;\hat{1}_{2N_0\times 2N_0}\;. && (4.15)
\end{aligned}$$

It remains to expand the actions $\mathcal{A}_{DET}[\hat{T},\langle\hat{\not{\mathcal{V}}}\rangle_{(3.59)};\hat{\mathcal{J}}]$, $\mathcal{A}_{J_\psi}[\hat{T},\langle\hat{\not{\mathcal{V}}}\rangle_{(3.59)};\hat{\mathcal{J}}]$ in terms of the gradient operator $\Delta\langle\hat{\mathcal{H}}\rangle_{(3.59)}=\hat{T}^{-1}\langle\hat{\mathcal{H}}\rangle_{(3.59)}\hat{T}-\langle\hat{\mathcal{H}}\rangle_{(3.59)}$ with mean field solution $\langle\hat{\not{\mathcal{V}}}(x_p)\rangle_{(3.59)}$ of (3.59) which is also applied for the anomalous doubled propagator $\langle\hat{\mathcal{H}}\rangle_{(3.59)}^{-\mathbf{1}}$

$$\begin{aligned}
\Big\langle\hat{\mathcal{H}}_{N;M}^{ba}(y_q,x_p)\Big\rangle_{(3.59)} &= \delta^{(4)}(y_q-x_p)\;\eta_q\;\delta_{qp}\begin{pmatrix}\big\langle\hat{H}_{N;M}(x_p)\big\rangle_{(3.59)} & \\ & \big\langle\hat{H}_{N;M}^{T}(x_p)\big\rangle_{(3.59)}\end{pmatrix}^{ba}=\delta^{(4)}(y_q-x_p)\;\eta_q\;\delta_{qp}\times\\
&\times\;\;\Big[\hat{S}\Big(\hat{\mathcal{B}}\;\hat{\Gamma}^{\mu}\;\hat{\partial}_{p,\mu}+\imath\;\hat{\mathcal{B}}\;\hat{\Gamma}^{\mu}\;\hat{\mathfrak{T}}_{\alpha}\;\big\langle\mathcal{V}_{\alpha;\mu}(x_p)\big\rangle_{(3.59)}+\hat{\mathcal{B}}\;\hat{M}\Big)-\imath\;\varepsilon_p\;\hat{1}_{2N_0\times 2N_0}\Big]\;; && (4.16)\\
\big\langle\hat{H}(x_p)\big\rangle_{(3.59)} &= \Big[\hat{\beta}\Big(\hat{\not{\partial}}_p+\imath\,\big\langle\hat{\not{\mathcal{V}}}(x_p)\big\rangle_{(3.59)}-\imath\,\hat{\varepsilon}_p+\hat{m}\Big)\Big]\;;\;(\hat{\varepsilon}_p=\hat{\beta}\;\varepsilon_p=\hat{\beta}\;\eta_p\;\varepsilon_+\;;\;\varepsilon_+>0)\;; && (4.17)\\
&= \hat{\beta}\hat{\gamma}^{\mu}\;\hat{\partial}_{p,\mu}+\imath\,\hat{\beta}\hat{\gamma}^{\mu}\;\hat{t}_{\alpha}\;\big\langle\mathcal{V}_{\alpha}^{\mu}(x_p)\big\rangle_{(3.59)}+\hat{\beta}\;\hat{m}-\imath\;\varepsilon_p\;\hat{1}_{N_0\times N_0}\;;\\
\big\langle\hat{H}^{T}(x_p)\big\rangle_{(3.59)} &= \Big[\hat{\beta}\Big(\hat{\not{\partial}}_p+\imath\,\big\langle\hat{\not{\mathcal{V}}}(x_p)\big\rangle_{(3.59)}-\imath\,\hat{\varepsilon}_p+\hat{m}\Big)\Big]^{T} && (4.18)\\
&= -(\,\hat{\beta}\hat{\gamma}^{\mu}\,)^{T}\;\hat{\partial}_{p,\mu}-\imath\;(\,\hat{\beta}\hat{\gamma}^{\mu}\,)^{T}\;(-\hat{t}_{\alpha}^{T})\;\big\langle\mathcal{V}_{\alpha}^{\mu}(x_p)\big\rangle_{(3.59)}-\big(\hat{\beta}\;(-\hat{m})\,\big)^{T}-\imath\;\varepsilon_p\;\hat{1}_{N_0\times N_0}\;;\\
\Big\langle\hat{\mathcal{H}}_{N;M}^{ba}(y_q,x_p)\Big\rangle_{(3.59)}^{-\mathbf{1}} &= \delta^{(4)}(y_q-x_p)\;\eta_q\;\delta_{qp}\begin{pmatrix}\big\langle\hat{H}_{N;M}(x_p)\big\rangle_{(3.59)}^{-\mathbf{1}} & \\ & \big\langle\hat{H}_{N;M}^{T}(x_p)\big\rangle_{(3.59)}^{-\mathbf{1}}\end{pmatrix}^{ba}; && (4.19)\\
\big\langle\hat{H}(x_p)\big\rangle_{(3.59)}^{-\mathbf{1}} &= \Big[\hat{\beta}\Big(\hat{\not{\partial}}_p+\imath\,\big\langle\hat{\not{\mathcal{V}}}(x_p)\big\rangle_{(3.59)}-\imath\,\hat{\varepsilon}_p+\hat{m}\Big)\Big]^{-\mathbf{1}}\;;\;(\hat{\varepsilon}_p=\hat{\beta}\;\varepsilon_p=\hat{\beta}\;\eta_p\;\varepsilon_+\;;\;\varepsilon_+>0)\;; && (4.20)
\end{aligned}$$

$$
\begin{aligned}
&= \Big[\hat{\beta}\hat{\gamma}^{\mu}\,\hat{\partial}_{p,\mu} + \imath\,\hat{\beta}\hat{\gamma}^{\mu}\,\hat{t}_{\alpha}\,\big\langle\mathcal{V}_{\alpha}^{\mu}(x_p)\big\rangle_{(3.59)} + \hat{\beta}\,\hat{m} - \imath\,\varepsilon_p\,\hat{1}_{N_0\times N_0}\Big]^{\boldsymbol{-1}}\;; \\
\big\langle\hat{H}^T(x_p)\big\rangle_{(3.59)}^{\boldsymbol{-1}} &= \Big[\hat{\beta}\Big(\hat{\not{\partial}}_p + \imath\,\langle\hat{\not{\mathcal{V}}}(x_p)\rangle_{(3.59)} - \imath\,\hat{\varepsilon}_p + \hat{m}\Big)\Big]^{\boldsymbol{T;-1}} \qquad (4.21)\\
&= \Big[-(\hat{\beta}\hat{\gamma}^{\mu})^T\,\hat{\partial}_{p,\mu} - \imath\,(\hat{\beta}\hat{\gamma}^{\mu})^T\,(-\hat{t}_{\alpha}^T)\,\big\langle\mathcal{V}_{\alpha}^{\mu}(x_p)\big\rangle_{(3.59)} - \big(\hat{\beta}\,(-\hat{m})\big)^T - \imath\,\varepsilon_p\,\hat{1}_{N_0\times N_0}\Big]^{\boldsymbol{-1}}\;.
\end{aligned}
$$

The relevant path integral (3.116) is thereby reduced to the relevant actions $\mathcal{A}_{DET}[\hat{T},\langle\hat{\not{\mathcal{V}}}\rangle_{(3.59)};\hat{\mathfrak{J}}]$, $\mathcal{A}_{J_\psi}[\hat{T},\langle\hat{\not{\mathcal{V}}}\rangle_{(3.59)};\hat{\mathfrak{J}}]$ of coset matrices $\hat{T}(x_p)=\exp\{-\hat{Y}(x_p)\}$ which are combined to the path integral $Z[\hat{\mathfrak{J}},J_\psi,\hat{J}_{\psi\psi};\langle\hat{\not{\mathcal{V}}}\rangle_{(3.59)};\hat{T}]$ (4.22) with averaged action $\langle Z_{\hat{J}_{\psi\psi}}[\hat{T}]\rangle_{(3.59)}$ for the initial configuration of BCS terms at times $t_{p=\pm}\to-\infty$. We neglect the detailed phase transition for the creation of coherent BCS terms from incoherent initial conditions which involves a detailed dependence of experimental parameters and temperature for the initial configuration of the nucleus; thus we simply set an initial configuration at intermediate time $0>t_{p=\pm}\gg-\infty$ by choosing a definite coset matrix $\hat{T}(x_p)=\exp\{-\hat{Y}(x_p)\}$

$$
\begin{aligned}
Z[\hat{\mathfrak{J}},J_\psi,\hat{J}_{\psi\psi};\langle\hat{\not{\mathcal{V}}}\rangle_{(3.59)};\hat{T}] &= \int d[\hat{T}^{-1}(x_p)\,d\hat{T}(x_p)]\;\;\Big\langle Z_{\hat{J}_{\psi\psi}}[\hat{T}]\Big\rangle_{(3.59)}\times \qquad (4.22)\\
&\times\;\exp\Big\{\mathcal{A}_{DET}[\hat{T},\langle\hat{\not{\mathcal{V}}}\rangle_{(3.59)};\hat{\mathfrak{J}}]\Big\}\times\exp\Big\{\imath\,\mathcal{A}_{J_\psi}[\hat{T},\langle\hat{\not{\mathcal{V}}}\rangle_{(3.59)};\hat{\mathfrak{J}}]\Big\}\;.
\end{aligned}
$$

4.2 Infinite order gradient expansion of logarithmic and inverted operators

The actions $\mathcal{A}_{DET}[\hat{T},\langle\hat{\not{\mathcal{V}}}\rangle_{(3.59)};\hat{\mathfrak{J}}]$, $\mathcal{A}_{J_\psi}[\hat{T},\langle\hat{\not{\mathcal{V}}}\rangle_{(3.59)};\hat{\mathfrak{J}}]$ (4.2,4.3) of (4.1-4.5) are transformed to relation (4.23) as the remaining path integral of BCS quark pairs within the coset decomposition $\mathrm{SO}(N_0,N_0)\,/\,\mathrm{U}(N_0)\otimes\mathrm{U}(N_0)$, $(N_0=(N_f=2)\times4_{\hat{\gamma}}\times(N_c=3)=24)$. We abbreviate (4.23) in terms of anomalous doubled Hilbert space states for the representation of coset operators, as e. g. $\hat{T}(x_p)=\exp\{-\hat{Y}(x_p)\}$ with coset generator $\hat{Y}(x_p)$ of $\mathrm{so}(N_0,N_0)\,/\,\mathrm{u}(N_0)$. (The reader is referred to appendix A for the important specification of doubled Hilbert space states with linear and anti-linear representations following from the dyadic product of anomalous doubled Fermi fields).

$$
\begin{aligned}
&Z[\hat{\mathfrak{J}},J_\psi,\hat{J}_{\psi\psi};\langle\hat{\not{\mathcal{V}}}\rangle_{(3.59)};\hat{T}] = \int d[\hat{T}^{-1}(x_p)\,d\hat{T}(x_p)]\;\;\Big\langle Z_{\hat{J}_{\psi\psi}}[\hat{T}]\Big\rangle_{(3.59)}\times \qquad (4.23)\\
&\times\;\exp\Big\{\frac{1}{2}\underset{\int_C d^4x_p\,\eta_p}{\overset{a(=1,2)}{\mathrm{TR}}}\;\underset{N_f,\hat{\gamma}_{mn}^{(\mu)},N_c}{\mathfrak{tr}}\Big(\ln\Big[\hat{1}+\Big(\Delta\langle\hat{\mathcal{H}}\rangle_{(3.59)}+\widetilde{\mathfrak{J}}(\hat{T}^{-1},\hat{T})\Big)\Big\langle\hat{\mathcal{H}}\Big\rangle_{(3.59)}^{\boldsymbol{-1}}\Big]\Big)\Big\}\times\\
&\times\;\exp\Big\{\frac{\imath}{2}\Big\langle\widehat{J_\psi}\Big|\hat{\eta}\,\hat{I}\Big[\hat{T}\,\langle\hat{\mathcal{H}}\rangle_{(3.59)}^{\boldsymbol{-1}}\Big(\hat{1}+\Big(\Delta\langle\hat{\mathcal{H}}\rangle_{(3.59)}+\widetilde{\mathfrak{J}}(\hat{T}^{-1},\hat{T})\Big)\Big\langle\hat{\mathcal{H}}\Big\rangle_{(3.59)}^{\boldsymbol{-1}}\Big)^{\boldsymbol{-1}}\hat{T}^{-1}-\Big\langle\hat{\mathcal{H}}\Big\rangle_{(3.59)}^{\boldsymbol{-1}}\Big]\hat{I}\,\hat{\eta}\Big|\widehat{J_\psi}\Big\rangle\Big\}\;.
\end{aligned}
$$

We have already defined the 'relative' gradient operator $\Delta\langle\hat{\mathcal{H}}\rangle_{(3.59)}$ (4.24) in (4.5); it is determined by the coset matrix weighted '$\hat{T}^{-1}\ldots\hat{T}$', anomalous doubled mean field operator $\langle\hat{\mathcal{H}}\rangle_{(3.59)}$ (see Eq. (4.1)) relative to its own eigenvalue spectrum and basis so that one has to subtract the mean field part $\langle\hat{\mathcal{H}}\rangle_{(3.59)}$ from the coset matrix weighted part $\hat{T}^{-1}\,\langle\hat{\mathcal{H}}\rangle_{(3.59)}\,\hat{T}$. Therefore, we have added and subtracted the anomalous doubled, mean field, one-particle operator $\langle\hat{\mathcal{H}}\rangle_{(3.59)}$ in $\langle\hat{\mathcal{O}}_{N;M}^{ba}(y_q,x_p)\rangle_{(3.59)}$ (4.4,4.5) so that one obtains in combination with $-\ln\langle\hat{\mathcal{H}}\rangle_{(3.59)}$ and $-\langle\hat{\mathcal{H}}\rangle_{(3.59)}^{\boldsymbol{-1}}$ within $\mathcal{A}_{DET}[\hat{T},\langle\hat{\not{\mathcal{V}}}\rangle_{(3.59)};\hat{\mathfrak{J}}]$ (4.2) and $\mathcal{A}_{J_\psi}[\hat{T},\langle\hat{\not{\mathcal{V}}}\rangle_{(3.59)};\hat{\mathfrak{J}}]$ (4.3) the path integral (4.23) specified by the operator $\hat{1}+\Delta\langle\hat{\mathcal{H}}\rangle_{(3.59)}\,\langle\hat{\mathcal{H}}\rangle_{(3.59)}^{\boldsymbol{-1}}$ (4.25). Aside from the source term $\widetilde{\mathfrak{J}}(\hat{T}^{-1},\hat{T})\,\langle\hat{\mathcal{H}}\rangle_{(3.59)}^{\boldsymbol{-1}}$ (4.26), this important operator part simplifies to the combination $\hat{T}^{-1}\,\langle\hat{\mathcal{H}}\rangle_{(3.59)}\,\hat{T}\,\langle\hat{\mathcal{H}}\rangle_{(3.59)}^{\boldsymbol{-1}}$ (4.25)

$$
\begin{aligned}
\Delta\big\langle\hat{\mathcal{H}}\big\rangle_{(3.59)} &= \hat{T}^{-1}\;\big\langle\hat{\mathcal{H}}\big\rangle_{(3.59)}\hat{T}-\big\langle\hat{\mathcal{H}}\big\rangle_{(3.59)}\;; \qquad (4.24)\\
\hat{1}+\Delta\big\langle\hat{\mathcal{H}}\big\rangle_{(3.59)}\,\big\langle\hat{\mathcal{H}}\big\rangle_{(3.59)}^{\boldsymbol{-1}} &= \hat{1}+\Big(\hat{T}^{-1}\;\big\langle\hat{\mathcal{H}}\big\rangle_{(3.59)}\hat{T}-\big\langle\hat{\mathcal{H}}\big\rangle_{(3.59)}\Big)\,\big\langle\hat{\mathcal{H}}\big\rangle_{(3.59)}^{\boldsymbol{-1}} \qquad (4.25)\\
&= \hat{T}^{-1}\;\big\langle\hat{\mathcal{H}}\big\rangle_{(3.59)}\hat{T}\;\big\langle\hat{\mathcal{H}}\big\rangle_{(3.59)}^{\boldsymbol{-1}}\;;
\end{aligned}
$$

$$\widetilde{\mathfrak{J}}(\hat{T}^{-1},\hat{T}) \;=\; \hat{T}^{-1}\,\hat{I}\,\hat{S}\,\hat{\eta}\,\frac{\hat{\mathfrak{J}}}{\mathcal{N}}\,\hat{\eta}\,\hat{S}\,\hat{I}\,\hat{T}\,. \tag{4.26}$$

However, as we try to reduce the lastly occurring gradient operator combination (4.25) to lowest order derivatives following from $\hat{T}^{-1}\,\langle\hat{\mathcal{H}}\rangle_{(3.59)}\,\hat{T}$, one has also to take into account the propagation of $\langle\hat{\mathcal{H}}\rangle^{-\mathbf{1}}_{(3.59)}$ which is weighted by $\hat{T}\,\langle\hat{\mathcal{H}}\rangle^{-\mathbf{1}}_{(3.59)}\,\hat{T}^{-1}$ according to the trace operations in (4.23). If we restrict to gradient operators up to order of four for slowly varying coset matrices in $\hat{T}^{-1}\,\langle\hat{\mathcal{H}}\rangle_{(3.59)}\,\hat{T}$ of a small momentum expansion ('Derrick's theorem' [13]), one unintentionally causes strongly varying fields $\hat{T}\,\langle\hat{\mathcal{H}}\rangle^{-\mathbf{1}}_{(3.59)}\,\hat{T}^{-1}$ from the inverse mean field operator $\langle\hat{\mathcal{H}}\rangle^{-\mathbf{1}}_{(3.59)}$. One can emphasize this point by a gauge transformation of the coset matrix $\hat{T}(x_p)\to\hat{T}_{\hat{\mathfrak{W}}}(x_p)$ so that the mean field operator $\langle\hat{\mathcal{H}}\rangle_{(3.59)}$ is altered to $\langle\hat{\mathcal{H}}_{\hat{\mathfrak{W}}}\rangle_{(3.59)}$ consisting only of pure gradient terms without any potential parts as $\langle\hat{\mathcal{V}}\rangle_{(3.59)}$ (compare section 4.3). In addition we alternatively suggest the exponential integral representations of the logarithm (4.27) and of the inverse (4.28) for the operator $\hat{\mathfrak{O}}_{\widetilde{\mathfrak{J}}}$ (4.29) so that one obtains a meaningful expansion and convergence with $1/n!$ instead of the reciprocal integer numbers of a logarithmic expansion [14]

$$\big(\ln\hat{\mathfrak{O}}\big) \;=\; \bigg(\int_0^{+\infty}dv\quad\frac{\exp\{-v\,\hat{1}\}-\exp\{-v\,\hat{\mathfrak{O}}\}}{v}\bigg)\,; \tag{4.27}$$

$$\big(\hat{\mathfrak{O}}^{-\mathbf{1}}\big) \;=\; \bigg(\int_0^{+\infty}dv\quad\exp\{-v\,\hat{\mathfrak{O}}\}\bigg)\,; \tag{4.28}$$

$$\begin{aligned}\hat{\mathfrak{O}}_{\widetilde{\mathfrak{J}}} &= \bigg(\hat{1}+\Big(\Delta\langle\hat{\mathcal{H}}\rangle_{(3.59)}+\widetilde{\mathfrak{J}}(\hat{T}^{-1},\hat{T})\Big)\,\Big\langle\hat{\mathcal{H}}\Big\rangle^{-\mathbf{1}}_{(3.59)}\bigg) \\ &= \bigg(\hat{T}^{-1}\;\langle\hat{\mathcal{H}}\;\rangle_{(3.59)}\hat{T}\;\langle\hat{\mathcal{H}}\rangle^{-\mathbf{1}}_{(3.59)}+\widetilde{\mathfrak{J}}(\hat{T}^{-1},\hat{T})\;\langle\hat{\mathcal{H}}\rangle^{-\mathbf{1}}_{(3.59)}\bigg)\,.\end{aligned} \tag{4.29}$$

In consequence one inserts Eqs. (4.27-4.29) into the path integral (4.23) with 'relative' gradient operator and source term (4.24-4.26) so that we achieve for the actions $\mathcal{A}_{DET}[\hat{T},\langle\hat{\mathcal{V}}\rangle_{(3.59)};\hat{\mathfrak{J}}]$, $\mathcal{A}_{J_\psi}[\hat{T},\langle\hat{\mathcal{V}}\rangle_{(3.59)};\hat{\mathfrak{J}}]$ (4.2,4.3) in (4.23,4.22) the exponential integral representations (4.30,4.31) with integration variable $v\in[0,\infty)$

$$\begin{aligned}\mathcal{A}_{DET}[\hat{T},\langle\hat{\mathcal{V}}\rangle_{(3.59)};\hat{\mathfrak{J}}] &= \frac{1}{2}\underset{\int_C d^4x_p\;\eta_p}{\overset{a(=1,2)}{\mathrm{TR}}}\;\underset{N_f,\hat{\gamma}^{(\mu)}_{mn},N_c}{\mathfrak{tr}}\bigg(\ln\Big[\hat{1}+\Big(\Delta\langle\hat{\mathcal{H}}\rangle_{(3.59)}+\widetilde{\mathfrak{J}}(\hat{T}^{-1},\hat{T})\Big)\,\Big\langle\hat{\mathcal{H}}\Big\rangle^{-\mathbf{1}}_{(3.59)}\Big]\bigg) \\ &= \frac{1}{2}\int_0^{+\infty}dv\;\;\exp\{-v\}\;\underset{\int_C d^4x_p\;\eta_p}{\overset{a(=1,2)}{\mathrm{TR}}}\;\underset{N_f,\hat{\gamma}^{(\mu)}_{mn},N_c}{\mathfrak{tr}}\left[\frac{\hat{1}-\exp\Big\{-v\,\Big(\Delta\langle\hat{\mathcal{H}}\rangle_{(3.59)}+\widetilde{\mathfrak{J}}(\hat{T}^{-1},\hat{T})\Big)\,\Big\langle\hat{\mathcal{H}}\Big\rangle^{-\mathbf{1}}_{(3.59)}\Big\}}{v}\right];\end{aligned} \tag{4.30}$$

$$\begin{aligned}&\imath\,\mathcal{A}_{J_\psi}[\hat{T},\langle\hat{\mathcal{V}}\rangle_{(3.59)};\hat{\mathfrak{J}}] = \\ &= \frac{\imath}{2}\Big\langle\widehat{J_\psi}\Big|\hat{\eta}\,\hat{I}\Big[\hat{T}\,\langle\hat{\mathcal{H}}\rangle^{-\mathbf{1}}_{(3.59)}\Big(\hat{1}+\Big(\Delta\langle\hat{\mathcal{H}}\rangle_{(3.59)}+\widetilde{\mathfrak{J}}(\hat{T}^{-1},\hat{T})\Big)\,\Big\langle\hat{\mathcal{H}}\Big\rangle^{-\mathbf{1}}_{(3.59)}\Big)^{-\mathbf{1}}\,\hat{T}^{-1}-\Big\langle\hat{\mathcal{H}}\Big\rangle^{-\mathbf{1}}_{(3.59)}\Big]\hat{I}\,\hat{\eta}\Big|\widehat{J_\psi}\Big\rangle \\ &= \frac{\imath}{2}\int_0^{+\infty}dv\;\exp\{-v\}\;\Big\langle\widehat{J_\psi}\Big|\hat{\eta}\,\hat{I}\Big[\hat{T}\,\Big\langle\hat{\mathcal{H}}\Big\rangle^{-\mathbf{1}}_{(3.59)}\exp\Big\{-v\,\Big(\Delta\langle\hat{\mathcal{H}}\rangle_{(3.59)}+\widetilde{\mathfrak{J}}(\hat{T}^{-1},\hat{T})\Big)\,\Big\langle\hat{\mathcal{H}}\Big\rangle^{-\mathbf{1}}_{(3.59)}\Big\}\,\hat{T}^{-1}\,\hat{I}\,\hat{\eta}\Big|\widehat{J_\psi}\Big\rangle+ \\ &- \frac{\imath}{2}\Big\langle\widehat{J_\psi}\Big|\hat{\eta}\,\hat{I}\,\Big\langle\hat{\mathcal{H}}\Big\rangle^{-\mathbf{1}}_{(3.59)}\,\hat{I}\,\hat{\eta}\Big|\widehat{J_\psi}\Big\rangle\,.\end{aligned} \tag{4.31}$$

As one applies relations (4.24,4.25) for the 'relative' gradient operator $\Delta\langle\hat{\mathcal{H}}\rangle_{(3.59)}$ in the case of a vanishing source term $\widetilde{\mathfrak{J}}(\hat{T}^{-1},\hat{T})$, we accomplish the exponential integral representations (4.32,4.33) for $\mathcal{A}_{DET}[\hat{T},\langle\hat{\mathcal{V}}\rangle_{(3.59)};\hat{\mathfrak{J}}\equiv 0]$ and for $\mathcal{A}_{J_\psi}[\hat{T},\langle\hat{\mathcal{V}}\rangle_{(3.59)};\hat{\mathfrak{J}}\equiv 0]$ with the exponent of the already described and composed operator $\hat{T}^{-1}\;\langle\hat{\mathcal{H}}\rangle_{(3.59)}\;\hat{T}\;\langle\hat{\mathcal{H}}\rangle^{-\mathbf{1}}_{(3.59)}$ (4.25). Since the latter operator has neither a valid small, nor large momentum expansion, the exponentials in (4.32,4.33)

give a meaningful representation for the previous form (4.2,4.3) of actions within (4.1,4.4) or for (4.23) with 'relative' gradients (4.24,4.25)

$$\mathcal{A}_{DET}[\hat{T},\langle\hat{\mathcal{V}}\rangle_{(3.59)};\hat{\mathcal{J}}\equiv 0]= \tag{4.32}$$
$$=\frac{1}{2}\int_0^{+\infty}dv\;\underset{\int_C d^4x_p\;\eta_p}{\overset{a(=1,2)}{\mathrm{TR}}}\;\underset{N_f,\hat{\gamma}_{mn}^{(\mu)},N_c}{\mathfrak{tr}}\left[\frac{\exp\{-v\,\hat{1}\}-\exp\Big\{-v\,\hat{T}^{-1}\langle\hat{\mathcal{H}}\rangle_{(3.59)}\hat{T}\langle\hat{\mathcal{H}}\rangle^{-\mathbf{1}}_{(3.59)}\Big\}}{v}\right];$$

$$\imath\,\mathcal{A}_{J_\psi}[\hat{T},\langle\hat{\mathcal{V}}\rangle_{(3.59)};\hat{\mathcal{J}}\equiv 0]= \tag{4.33}$$
$$=\frac{\imath}{2}\int_0^{+\infty}dv\Big\langle\widehat{J_\psi}\Big|\hat{\eta}\,\hat{I}\hat{T}\,\Big\langle\hat{\mathcal{H}}\Big\rangle^{-\mathbf{1}}_{(3.59)}\exp\Big\{-v\,\hat{T}^{-1}\langle\hat{\mathcal{H}}\rangle_{(3.59)}\hat{T}\langle\hat{\mathcal{H}}\rangle^{-\mathbf{1}}_{(3.59)}\Big\}\,\hat{T}^{-1}\,\hat{I}\,\hat{\eta}\Big|\widehat{J_\psi}\Big\rangle-\frac{\imath}{2}\Big\langle\widehat{J_\psi}\Big|\hat{\eta}\,\hat{I}\,\Big\langle\hat{\mathcal{H}}\Big\rangle^{-\mathbf{1}}_{(3.59)}\,\hat{I}\,\hat{\eta}\Big|\widehat{J_\psi}\Big\rangle\,.$$

The representations (4.32,4.33) for the actions $\mathcal{A}_{DET}[\hat{T},\langle\hat{\mathcal{V}}\rangle_{(3.59)};\hat{\mathcal{J}}\equiv 0]$, $\mathcal{A}_{J_\psi}[\hat{T},\langle\hat{\mathcal{V}}\rangle_{(3.59)};\hat{\mathcal{J}}\equiv 0]$ allow a straightforward calculation of observables, especially after a gauge transformation to pure gradient terms $\langle\hat{\mathcal{H}}\rangle_{(3.59)}\to\langle\hat{\mathcal{H}}_{\hat{\mathfrak{W}}}\rangle_{(3.59)}$, $\hat{T}(x_p)\to\hat{T}_{\hat{\mathfrak{W}}}(x_p)$. In order to compute correlation functions, we consider again the operator $\hat{\mathfrak{O}}_{\widetilde{\mathcal{J}}}$ (4.34), but under inclusion of the source term $\widetilde{\mathcal{J}}(\hat{T}^{-1},\hat{T})$. One can track one-point or two-point correlation functions of the original Grassmann-valued fields $\psi_M(x_p)$ in the orginal QCD-type path integral (2.25-2.27) by subsequent differentiation of $\hat{\mathcal{J}}^{ba}_{N;M}(y_q,x_p)$ (2.22-2.24) to the corresponding observables in terms of the coset matrices from the actions $\mathcal{A}_{DET}[\hat{T},\langle\hat{\mathcal{V}}\rangle_{(3.59)};\hat{\mathcal{J}}]$, $\mathcal{A}_{J_\psi}[\hat{T},\langle\hat{\mathcal{V}}\rangle_{(3.59)};\hat{\mathcal{J}}]$. Therefore, we perform variations $\delta_{\hat{\mathcal{J}}(\mathrm{arg.1})}$ (4.35), $\delta_{\hat{\mathcal{J}}(\mathrm{arg.2})}$ (4.36) of the particular operator $\hat{\mathfrak{O}}_{\widetilde{\mathcal{J}}}$ (4.34) with arguments '(arg.1)', '(arg.2)' (4.37,4.38) for one-point and two-point correlation functions, respectively

$$\begin{aligned}
\hat{\mathfrak{O}}_{\widetilde{\mathcal{J}}} &= \hat{1}+\Delta\langle\hat{\mathcal{H}}\rangle_{(3.59)}\;\langle\hat{\mathcal{H}}\rangle^{-\mathbf{1}}_{(3.59)}+\widetilde{\mathcal{J}}(\hat{T}^{-1},\hat{T})\;\langle\hat{\mathcal{H}}\rangle^{-\mathbf{1}}_{(3.59)} && (4.34)\\
&= \hat{1}+\Big(\hat{T}^{-1}\;\langle\hat{\mathcal{H}}\;\;\rangle_{(3.59)}\hat{T}-\langle\hat{\mathcal{H}}\rangle_{(3.59)}\Big)\;\langle\hat{\mathcal{H}}\rangle^{-\mathbf{1}}_{(3.59)}+\widetilde{\mathcal{J}}(\hat{T}^{-1},\hat{T})\;\langle\hat{\mathcal{H}}\rangle^{-\mathbf{1}}_{(3.59)}\\
&= \Big(\hat{T}^{-1}\;\langle\hat{\mathcal{H}}\;\;\rangle_{(3.59)}\hat{T}\;\;\langle\hat{\mathcal{H}}\rangle^{-\mathbf{1}}_{(3.59)}+\widetilde{\mathcal{J}}(\hat{T}^{-1},\hat{T})\;\langle\hat{\mathcal{H}}\rangle^{-\mathbf{1}}_{(3.59)}\Big)\,;\\
\delta_{\hat{\mathcal{J}}(\mathrm{arg.1})} &= \delta_{(\mathrm{arg.1})}\hat{\mathfrak{O}}=\hat{T}^{-1}\;\hat{I}\;\hat{S}\;\hat{\eta}\;\frac{\delta\hat{\mathcal{J}}(\mathrm{arg.1})}{\mathcal{N}}\;\hat{\eta}\;\hat{S}\;\hat{I}\;\hat{T}\;\langle\hat{\mathcal{H}}\rangle^{-\mathbf{1}}_{(3.59)}\,; && (4.35)\\
\delta_{\hat{\mathcal{J}}(\mathrm{arg.2})} &= \delta_{(\mathrm{arg.2})}\hat{\mathfrak{O}}=\hat{T}^{-1}\;\hat{I}\;\hat{S}\;\hat{\eta}\;\frac{\delta\hat{\mathcal{J}}(\mathrm{arg.2})}{\mathcal{N}}\;\hat{\eta}\;\hat{S}\;\hat{I}\;\hat{T}\;\langle\hat{\mathcal{H}}\rangle^{-\mathbf{1}}_{(3.59)}\,; && (4.36)\\
\delta\hat{\mathcal{J}}(\mathrm{arg.1}) &:= \delta\hat{\mathcal{J}}^{a_1b_1}_{M_1;N_1}(x^{(1)}_{p_1},y^{(1)}_{q_1})\,; && (4.37)\\
\delta\hat{\mathcal{J}}(\mathrm{arg.2}) &:= \delta\hat{\mathcal{J}}^{a_2b_2}_{M_2;N_2}(x^{(2)}_{p_2},y^{(2)}_{q_2})\,. && (4.38)
\end{aligned}$$

Suitable integral representations are used for the variation of the logarithmic and inverted operators of $\hat{\mathfrak{O}}$ which are transformed to exponential integral representations for a meaningful expansion and convergence according to the $1\,/\,n!$ reciprocal factorials [14]

$$\begin{aligned}
\delta\big(\ln\hat{\mathfrak{O}}\big) &= \left(\int_0^{+\infty}du\;\;\frac{\hat{1}}{\hat{1}\,u+\hat{\mathfrak{O}}}\;\big(\delta\hat{\mathfrak{O}}\big)\;\frac{\hat{1}}{\hat{1}\,u+\hat{\mathfrak{O}}}\right); && (4.39)\\
\delta\big(\hat{\mathfrak{O}}^{-\mathbf{1}}\big) &= -\Big(\hat{\mathfrak{O}}^{-\mathbf{1}}\;\big(\delta\hat{\mathfrak{O}}\big)\;\hat{\mathfrak{O}}^{-\mathbf{1}}\Big)\,; && (4.40)\\
\delta\big(\ln\hat{\mathfrak{O}}\big) &= \int_0^{+\infty}du\;dv_1\;dv_2\;\;\exp\{-u(v_1+v_2)\}\Big(\exp\{-v_1\,\hat{\mathfrak{O}}\}\;\big(\delta\hat{\mathfrak{O}}\big)\;\exp\{-v_2\,\hat{\mathfrak{O}}\}\Big)\,; && (4.41)\\
\delta\big(\hat{\mathfrak{O}}^{-\mathbf{1}}\big) &= -\int_0^{+\infty}dv_1\;dv_2\;\;\Big(\exp\{-v_1\,\hat{\mathfrak{O}}\}\;\big(\delta\hat{\mathfrak{O}}\big)\;\exp\{-v_2\,\hat{\mathfrak{O}}\}\Big)\,. && (4.42)
\end{aligned}$$

As we consider the rather involved appearing, first order variation $\delta_{\hat{\mathcal{J}}(\text{arg.1})}$ (4.35,4.37) of $\mathcal{A}_{DET}[\hat{T},\langle\hat{\mathcal{V}}\rangle_{(3.59)};\hat{\mathcal{J}}]$ and of $\mathcal{A}_{J_\psi}[\hat{T},\langle\hat{\mathcal{V}}\rangle_{(3.59)};\hat{\mathcal{J}}]$, we finally attain the one-point observable (4.45) from (4.43,4.44) which is mainly determined by exponential integrals of the operator $\hat{T}^{-1}\;\langle\hat{\mathcal{H}}\rangle_{(3.59)}\;\hat{T}\;\langle\hat{\mathcal{H}}\rangle^{-\mathbf{1}}_{(3.59)}$ (4.25) having neither a valid small nor large momentum expansion. It has to be noted that relation (4.45) comprises anomalous parts as $\langle\psi_N(y_q)\;\psi_M(x_p)\rangle$ with $b=2$, $a=1$ and as well density terms as $\langle\psi_N^*(y_q)\;\psi_M(x_p)\rangle$ by choosing the anomalous indices $b=1$, $a=1$. As one neglects the functional $\langle Z_{\hat{J}_{\psi\psi}}[\hat{T}]\rangle_{(3.59)}$ by taking appropriate initial conditions for $\hat{T}(x_p)$, one can accomplish the rather involved appearing path integral (4.45) for the one-point correlation function which becomes more accessible by transforming to the eigenbasis of $\langle\hat{\mathcal{H}}\rangle_{(3.59)}$. This is in particular applicable for the bulk of the nucleus where surface effects are negligible and where the mean field potential $\langle\hat{\mathcal{V}}\rangle_{(3.59)}$ is expected to have a constant value. However, surface effects can additionally be taken into account by appropriate Ward identities for the coset decomposition (cf appendix B)

$$\begin{aligned}
&\delta_{\hat{\mathcal{J}}(\text{arg.1})}\exp\Big\{\mathcal{A}_{DET}[\hat{T},\langle\hat{\mathcal{V}}\rangle_{(3.59)};\hat{\mathcal{J}}]\Big\}=\delta_{\hat{\mathcal{J}}(\text{arg.1})}\mathcal{A}_{DET}[\hat{T},\langle\hat{\mathcal{V}}\rangle_{(3.59)};\hat{\mathcal{J}}] \qquad (4.43)\\
&=\;\frac{1}{2}\underset{\int_C d^4x_p\;\eta_p}{\overset{a(=1,2)}{\mathrm{TR}}}\;\underset{N_f,\hat{\gamma}_{mn}^{(\mu)},N_c}{\mathfrak{tr}}\;\delta_{\hat{\mathcal{J}}(\text{arg.1})}\ln\Big(\underbrace{\hat{1}+\Delta\langle\hat{\mathcal{H}}\rangle_{(3.59)}\;\langle\hat{\mathcal{H}}\rangle^{-\mathbf{1}}_{(3.59)}}_{\hat{\mathfrak{S}}_{\widetilde{\mathcal{J}}=0}}+\widetilde{\mathcal{J}}(\hat{T}^{-1},\hat{T})\;\langle\hat{\mathcal{H}}\rangle^{-\mathbf{1}}_{(3.59)}\Big)\\
&=\;\frac{1}{2}\int_0^{+\infty}du\;dv_1\;dv_2\;\exp\Big\{-u\;\big(v_1+v_2\big)\Big\}\times\underset{\int_C d^4x_p\;\eta_p}{\overset{a(=1,2)}{\mathrm{TR}}}\;\underset{N_f,\hat{\gamma}_{mn}^{(\mu)},N_c}{\mathfrak{tr}}\Big[\exp\Big\{-v_1\;\hat{T}^{-1}\;\langle\hat{\mathcal{H}}\rangle_{(3.59)}\;\hat{T}\;\langle\hat{\mathcal{H}}\rangle^{-\mathbf{1}}_{(3.59)}\Big\}\times\\
&\times\;\hat{T}^{-1}\;\hat{I}\;\hat{S}\;\hat{\eta}\;\frac{\delta\hat{\mathcal{J}}(\text{arg.1})}{\mathcal{N}}\;\hat{\eta}\;\hat{S}\;\hat{I}\;\hat{T}\;\langle\hat{\mathcal{H}}\rangle^{-\mathbf{1}}_{(3.59)}\exp\Big\{-v_2\;\hat{T}^{-1}\;\langle\hat{\mathcal{H}}\rangle_{(3.59)}\;\hat{T}\;\langle\hat{\mathcal{H}}\rangle^{-\mathbf{1}}_{(3.59)}\Big\}\Big]\;;
\end{aligned}$$

$$\begin{aligned}
&\delta_{\hat{\mathcal{J}}(\text{arg.1})}\exp\Big\{\imath\,\mathcal{A}_{J_\psi}[\hat{T},\langle\hat{\mathcal{V}}\rangle_{(3.59)};\hat{\mathcal{J}}]\Big\}=\imath\;\delta_{\hat{\mathcal{J}}(\text{arg.1})}\mathcal{A}_{J_\psi}[\hat{T},\langle\hat{\mathcal{V}}\rangle_{(3.59)};\hat{\mathcal{J}}] \qquad (4.44)\\
&=\;\frac{\imath}{2}\Big\langle\widehat{J_\psi}\Big|\hat{\eta}\;\hat{I}\hat{T}\;\Big\langle\hat{\mathcal{H}}\Big\rangle^{-\mathbf{1}}_{(3.59)}\;\delta_{\hat{\mathcal{J}}(\text{arg.1})}\Big(\underbrace{\hat{1}+\Delta\langle\hat{\mathcal{H}}\rangle_{(3.59)}\;\langle\hat{\mathcal{H}}\rangle^{-\mathbf{1}}_{(3.59)}}_{\hat{\mathfrak{S}}_{\widetilde{\mathcal{J}}=0}}+\widetilde{\mathcal{J}}(\hat{T}^{-1},\hat{T})\;\langle\hat{\mathcal{H}}\rangle^{-\mathbf{1}}_{(3.59)}\Big)^{-\mathbf{1}}\;\hat{T}^{-1}\;\hat{I}\;\hat{\eta}\Big|\widehat{J_\psi}\Big\rangle\\
&=\;-\frac{\imath}{2}\int_0^{+\infty}dv_1\;dv_2\;\Big\langle\widehat{J_\psi}\Big|\hat{\eta}\;\hat{I}\hat{T}\;\Big\langle\hat{\mathcal{H}}\Big\rangle^{-\mathbf{1}}_{(3.59)}\;\exp\Big\{-v_1\;\hat{T}^{-1}\;\langle\hat{\mathcal{H}}\rangle_{(3.59)}\;\hat{T}\;\langle\hat{\mathcal{H}}\rangle^{-\mathbf{1}}_{(3.59)}\Big\}\times\\
&\times\;\hat{T}^{-1}\;\hat{I}\;\hat{S}\;\hat{\eta}\;\frac{\delta\hat{\mathcal{J}}(\text{arg.1})}{\mathcal{N}}\;\hat{\eta}\;\hat{S}\;\hat{I}\;\hat{T}\;\langle\hat{\mathcal{H}}\rangle^{-\mathbf{1}}_{(3.59)}\exp\Big\{-v_2\;\hat{T}^{-1}\;\langle\hat{\mathcal{H}}\rangle_{(3.59)}\;\hat{T}\;\langle\hat{\mathcal{H}}\rangle^{-\mathbf{1}}_{(3.59)}\Big\}\;\hat{T}^{-1}\;\hat{I}\;\hat{\eta}\Big|\widehat{J_\psi}\Big\rangle\\
&=\;\frac{\imath}{2}\int_0^{+\infty}dv_1\;dv_2\;\underset{\int_C d^4x_p\;\eta_p}{\overset{a(=1,2)}{\mathrm{TR}}}\;\underset{N_f,\hat{\gamma}_{mn}^{(\mu)},N_c}{\mathfrak{tr}}\Big[\exp\Big\{-v_1\;\hat{T}^{-1}\;\langle\hat{\mathcal{H}}\rangle_{(3.59)}\;\hat{T}\;\langle\hat{\mathcal{H}}\rangle^{-\mathbf{1}}_{(3.59)}\Big\}\times\\
&\times\;\hat{T}^{-1}\;\hat{I}\;\hat{S}\;\hat{\eta}\;\frac{\delta\hat{\mathcal{J}}(\text{arg.1})}{\mathcal{N}}\;\hat{\eta}\;\hat{S}\;\hat{I}\;\hat{T}\;\langle\hat{\mathcal{H}}\rangle^{-\mathbf{1}}_{(3.59)}\exp\Big\{-v_2\;\hat{T}^{-1}\;\langle\hat{\mathcal{H}}\rangle_{(3.59)}\;\hat{T}\;\langle\hat{\mathcal{H}}\rangle^{-\mathbf{1}}_{(3.59)}\Big\}\times\\
&\times\;\hat{T}^{-1}\;\hat{I}\;\hat{\eta}\Big|\widehat{J_\psi}\Big\rangle\otimes\Big\langle\widehat{J_\psi}\Big|\hat{\eta}\;\hat{I}\hat{T}\;\Big\langle\hat{\mathcal{H}}\Big\rangle^{-\mathbf{1}}_{(3.59)}\Big]\;.
\end{aligned}$$

Referring to the original QCD-type path integral with relations (2.22-2.27), we obtain for the one-point correlation $\langle\Psi_N^{\dagger,b}(y_q)\;\Psi_M^a(x_p)\rangle$ the following path integral in terms of BCS quark pairs within the coset generator $\hat{Y}(x_p)$ of the coset matrices $\hat{T}(x_p)=\exp\{-\hat{Y}(x_p)\}$

$$-\frac{\imath}{2}\Big\langle\Psi_N^{\dagger,b}(y_q)\;\Psi_M^a(x_p)\Big\rangle=\int d[\hat{T}^{-1}(x_p)\;d\hat{T}(x_p)]\quad\Big\langle Z_{\hat{J}_{\psi\psi}}[\hat{T}]\Big\rangle_{(3.59)}\times \qquad (4.45)$$

$$
\begin{aligned}
\times &\quad \exp\Bigg\{\frac{1}{2}\int_0^{+\infty} dv \quad \underset{\int_C d^4x_p\;\eta_p}{\overset{a(=1,2)}{\mathrm{TR}}} \;\underset{N_f,\hat{\gamma}_{mn}^{(\mu)},N_c}{\mathfrak{tr}}\Bigg[\frac{1}{v}\Big(e^{-v\,\hat{1}} - e^{-v\,\hat{T}^{-1}\langle\hat{\mathcal{H}}\rangle_{(3.59)}\hat{T}\langle\hat{\mathcal{H}}\rangle^{-\mathbf{1}}_{(3.59)}}\Big)\Bigg]\Bigg\}\times \\
\times &\quad \exp\Bigg\{\frac{\imath}{2}\int_0^{+\infty} dv\,\Big\langle\widehat{J_\psi}\Big|\hat{\eta}\,\hat{I}\Big(\hat{T}\,\langle\hat{\mathcal{H}}\rangle^{-\mathbf{1}}_{(3.59)}e^{-v\,\hat{T}^{-1}\langle\hat{\mathcal{H}}\rangle_{(3.59)}\hat{T}\langle\hat{\mathcal{H}}\rangle^{-\mathbf{1}}_{(3.59)}}\,\hat{T}^{-1} - \langle\hat{\mathcal{H}}\rangle^{-\mathbf{1}}_{(3.59)}\Big)\,\hat{I}\,\hat{\eta}\Big|\widehat{J_\psi}\Big\rangle\Bigg\}\times \\
\times &\quad \Bigg\{\frac{1}{2}\int_0^{+\infty} dv_1\,dv_2 \quad \underset{\int_C d^4x_p\;\eta_p}{\overset{a(=1,2)}{\mathrm{TR}}}\;\underset{N_f,\hat{\gamma}_{mn}^{(\mu)},N_c}{\mathfrak{tr}}\Bigg[\Bigg(e^{-v_1\,\hat{T}^{-1}\,\langle\hat{\mathcal{H}}\rangle_{(3.59)}\,\hat{T}\,\langle\hat{\mathcal{H}}\rangle^{-\mathbf{1}}_{(3.59)}}\Big|\widehat{y_q}\Big\rangle^{b'}_{N'}\;\hat{T}^{-1;b'b}_{N';N}(y_q)\,\hat{S}\,\hat{T}^{aa'}_{M;M'}(x_p)\;\times \\
\times &\quad {}^{a'}_{M'}\Big\langle\widehat{x_p}\Big|\langle\hat{\mathcal{H}}\rangle^{-\mathbf{1}}_{(3.59)}e^{-v_2\,\hat{T}^{-1}\,\langle\hat{\mathcal{H}}\rangle_{(3.59)}\,\hat{T}\,\langle\hat{\mathcal{H}}\rangle^{-\mathbf{1}}_{(3.59)}}\Bigg)\times \\
\times &\quad \Bigg(\int_0^{+\infty} du\;\hat{1}\;e^{-u\,(v_1+v_2)} + \imath\Big(\hat{T}^{-1}\,\hat{I}\,\hat{\eta}\Big|\widehat{J_\psi}\Big\rangle\otimes\Big\langle\widehat{J_\psi}\Big|\hat{\eta}\,\hat{I}\hat{T}\,\Big\langle\hat{\mathcal{H}}\Big\rangle^{-\mathbf{1}}_{(3.59)}\Big)\Bigg)\Bigg]\Bigg\}\,.
\end{aligned}
$$

Eventually, we also state the second order variations $\delta_{\hat{\mathfrak{J}}(\text{arg.2})}\delta_{\hat{\mathfrak{J}}(\text{arg.1})}$ of $\mathcal{A}_{DET}[\hat{T},\langle\hat{\mathcal{V}}\rangle_{(3.59)};\hat{\mathfrak{J}}]$ and $\mathcal{A}_{J_\psi}[\hat{T},\langle\hat{\mathcal{V}}\rangle_{(3.59)};\hat{\mathfrak{J}}]$ for two-point correlations so that it becomes possible to compute e. g. eigenvalue correlations of nuclei

$$
\begin{aligned}
&\delta_{\hat{\mathfrak{J}}(\text{arg.2})}\delta_{\hat{\mathfrak{J}}(\text{arg.1})}\exp\Big\{\mathcal{A}_{DET}[\hat{T},\langle\hat{\mathcal{V}}\rangle_{(3.59)};\hat{\mathfrak{J}}] + \imath\,\mathcal{A}_{J_\psi}[\hat{T},\langle\hat{\mathcal{V}}\rangle_{(3.59)};\hat{\mathfrak{J}}]\Big\} = \qquad (4.46) \\
= &\quad \frac{1}{2}\Bigg(\underset{\int_C d^4x_p\;\eta_p}{\overset{a(=1,2)}{\mathrm{TR}}}\;\underset{N_f,\hat{\gamma}_{mn}^{(\mu)},N_c}{\mathfrak{tr}}\delta_{\hat{\mathfrak{J}}(\text{arg.2})}\big(\ln\hat{\mathfrak{O}}_{\tilde{\mathfrak{J}}}\big) + \imath\big\langle\widehat{J_\psi}\big|\hat{\eta}\,\hat{I}\,\hat{T}\,\langle\hat{\mathcal{H}}\rangle^{-\mathbf{1}}_{(3.59)}\delta_{\hat{\mathfrak{J}}(\text{arg.2})}\big(\hat{\mathfrak{O}}_{\tilde{\mathfrak{J}}}\big)^{-\mathbf{1}}\,\hat{T}^{-1}\,\hat{I}\,\hat{\eta}\big|\widehat{J_\psi}\big\rangle\Bigg)\times \\
\times &\quad \frac{1}{2}\Bigg(\underset{\int_C d^4x_p\;\eta_p}{\overset{a(=1,2)}{\mathrm{TR}}}\;\underset{N_f,\hat{\gamma}_{mn}^{(\mu)},N_c}{\mathfrak{tr}}\delta_{\hat{\mathfrak{J}}(\text{arg.1})}\big(\ln\hat{\mathfrak{O}}_{\tilde{\mathfrak{J}}}\big) + \imath\big\langle\widehat{J_\psi}\big|\hat{\eta}\,\hat{I}\,\hat{T}\,\langle\hat{\mathcal{H}}\rangle^{-\mathbf{1}}_{(3.59)}\delta_{\hat{\mathfrak{J}}(\text{arg.1})}\big(\hat{\mathfrak{O}}_{\tilde{\mathfrak{J}}}\big)^{-\mathbf{1}}\,\hat{T}^{-1}\,\hat{I}\,\hat{\eta}\big|\widehat{J_\psi}\big\rangle\Bigg) + \\
- &\quad \frac{1}{2}\int_0^{+\infty} du \quad \underset{\int_C d^4x_p\;\eta_p}{\overset{a(=1,2)}{\mathrm{TR}}}\;\underset{N_f,\hat{\gamma}_{mn}^{(\mu)},N_c}{\mathfrak{tr}}\Bigg[\frac{1}{\hat{1}\,u+\hat{\mathfrak{O}}_{\tilde{\mathfrak{J}}}}\Bigg\{\big(\delta_{\hat{\mathfrak{J}}(\text{arg.2})}\hat{\mathfrak{O}}_{\tilde{\mathfrak{J}}}\big)\frac{1}{\hat{1}\,u+\hat{\mathfrak{O}}_{\tilde{\mathfrak{J}}}}\,\mathbf{,}\,\big(\delta_{\hat{\mathfrak{J}}(\text{arg.1})}\hat{\mathfrak{O}}_{\tilde{\mathfrak{J}}}\big)\frac{1}{\hat{1}\,u+\hat{\mathfrak{O}}_{\tilde{\mathfrak{J}}}}\Bigg\}_+\Bigg] + \\
+ &\quad \frac{\imath}{2}\Big\langle\widehat{J_\psi}\Big|\hat{\eta}\,\hat{I}\,\hat{T}\,\langle\hat{\mathcal{H}}\rangle^{-\mathbf{1}}_{(3.59)}\big(\hat{\mathfrak{O}}_{\tilde{\mathfrak{J}}}\big)^{-\mathbf{1}}\Big\{\big(\delta_{\hat{\mathfrak{J}}(\text{arg.2})}\hat{\mathfrak{O}}_{\tilde{\mathfrak{J}}}\big)\,\big(\hat{\mathfrak{O}}_{\tilde{\mathfrak{J}}}\big)^{-\mathbf{1}}\,\mathbf{,}\,\big(\delta_{\hat{\mathfrak{J}}(\text{arg.1})}\hat{\mathfrak{O}}_{\tilde{\mathfrak{J}}}\big)\,\big(\hat{\mathfrak{O}}_{\tilde{\mathfrak{J}}}\big)^{-\mathbf{1}}\Big\}_+\hat{T}^{-1}\,\hat{I}\,\hat{\eta}\Big|\widehat{J_\psi}\Big\rangle = \\
= &\quad -\frac{1}{4}\Bigg(\Big\langle\Psi^{\dagger,b_2}_{N_2}(y^{(2)}_{q_2})\;\Psi^{a_2}_{M_2}(x^{(2)}_{p_2})\Big\rangle\Big\langle\Psi^{\dagger,b_1}_{N_1}(y^{(1)}_{q_1})\;\Psi^{a_1}_{M_1}(x^{(1)}_{p_1})\Big\rangle + \Big\langle\Psi^{\dagger,b_2}_{N_2}(y^{(2)}_{q_2})\;\Psi^{a_2}_{M_2}(x^{(2)}_{p_2})\;\;\Psi^{\dagger,b_1}_{N_1}(y^{(1)}_{q_1})\;\Psi^{a_1}_{M_1}(x^{(1)}_{p_1})\Big\rangle\Bigg)\,.
\end{aligned}
$$

This straightforward, but involved appearing second order variation (4.46) of the source term $\tilde{\mathfrak{J}}(\hat{T}^{-1},\hat{T})$ (4.26) results into relation (4.47) for the two-point correlator which also considerably simplifies through the transformation from the coordinate spacetime representation of $\hat{T}(x_p)$ to the eigenbasis of the mean field operator $\langle\hat{\mathcal{H}}\rangle_{(3.59)}$ with mean field potential $\langle\hat{\mathcal{V}}\rangle_{(3.59)}$

$$
\begin{aligned}
&\Big\langle\Psi^{\dagger,b_2}_{N_2}(y^{(2)}_{q_2})\;\Psi^{a_2}_{M_2}(x^{(2)}_{p_2})\;\times\;\Psi^{\dagger,b_1}_{N_1}(y^{(1)}_{q_1})\;\Psi^{a_1}_{M_1}(x^{(1)}_{p_1})\Big\rangle - \Big\langle\Psi^{\dagger,b_2}_{N_2}(y^{(2)}_{q_2})\;\Psi^{a_2}_{M_2}(x^{(2)}_{p_2})\Big\rangle\times\Big\langle\Psi^{\dagger,b_1}_{N_1}(y^{(1)}_{q_1})\;\Psi^{a_1}_{M_1}(x^{(1)}_{p_1})\Big\rangle = \\
= &\quad 4\int d[\hat{T}^{-1}(x_p)\,d\hat{T}(x_p)]\quad\Big\langle Z_{\hat{J}_{\psi\psi}}[\hat{T}]\Big\rangle_{(3.59)}\;\times \qquad (4.47) \\
\times &\quad \exp\Bigg\{\frac{1}{2}\int_0^{+\infty} dv \quad \underset{\int_C d^4x_p\;\eta_p}{\overset{a(=1,2)}{\mathrm{TR}}}\;\underset{N_f,\hat{\gamma}_{mn}^{(\mu)},N_c}{\mathfrak{tr}}\Bigg[\frac{1}{v}\Big(e^{-v\,\hat{1}} - e^{-v\,\hat{T}^{-1}\langle\hat{\mathcal{H}}\rangle_{(3.59)}\hat{T}\langle\hat{\mathcal{H}}\rangle^{-\mathbf{1}}_{(3.59)}}\Big)\Bigg]\Bigg\}\times \\
\times &\quad \exp\Bigg\{\frac{\imath}{2}\int_0^{+\infty} dv\,\Big\langle\widehat{J_\psi}\Big|\hat{\eta}\,\hat{I}\Big(\hat{T}\,\langle\hat{\mathcal{H}}\rangle^{-\mathbf{1}}_{(3.59)}e^{-v\,\hat{T}^{-1}\langle\hat{\mathcal{H}}\rangle_{(3.59)}\hat{T}\langle\hat{\mathcal{H}}\rangle^{-\mathbf{1}}_{(3.59)}}\,\hat{T}^{-1} - \langle\hat{\mathcal{H}}\rangle^{-\mathbf{1}}_{(3.59)}\Big)\,\hat{I}\,\hat{\eta}\Big|\widehat{J_\psi}\Big\rangle\Bigg\}\times
\end{aligned}
$$

$$
\begin{aligned}
\times\;& \left\{\frac{1}{2}\int_0^{+\infty} dv_1\,dv_2\,dv_3 \;\;\underset{\int_C d^4x_p\;\eta_p}{\overset{a(=1,2)}{\mathrm{TR}}}\;\;\underset{N_f,\hat{\gamma}_{mn}^{(\mu)},N_c}{\mathfrak{tr}}\Bigg[\Bigg(\Bigg(\Big\langle e^{-v_3\,\hat{T}^{-1}\,\langle\hat{\mathcal{H}}\rangle_{(3.59)}\,\hat{T}\,\langle\hat{\mathcal{H}}\rangle^{-\mathbf{1}}_{(3.59)}}\Big|\widehat{y_{q_2}}^{(2)}\Big\rangle_{N_2'}^{b_2'}\times\right.\\
\times\;& \hat{T}^{-1;b_2'b_2}_{N_2';N_2}(y_{q_2}^{(2)})\;\hat{S}\;\hat{T}^{a_2a_2'}_{M_2;M_2'}(x_{p_2}^{(2)})\;{}^{a_2'}_{M_2'}\Big\langle\widehat{x_{p_2}^{(2)}}\Big|\langle\hat{\mathcal{H}}\rangle^{-\mathbf{1}}_{(3.59)}e^{-v_2\,\hat{T}^{-1}\,\langle\hat{\mathcal{H}}\rangle_{(3.59)}\,\hat{T}\,\langle\hat{\mathcal{H}}\rangle^{-\mathbf{1}}_{(3.59)}}\Big|\widehat{y_{q_1}}^{(1)}\Big\rangle_{N_1'}^{b_1'}\times\\
\times\;& \hat{T}^{-1;b_1'b_1}_{N_1';N_1}(y_{q_1}^{(1)})\;\hat{S}\;\hat{T}^{a_1a_1'}_{M_1;M_1'}(x_{p_1}^{(1)})\;{}^{a_1'}_{M_1'}\Big\langle\widehat{x_{p_1}^{(1)}}\Big|\langle\hat{\mathcal{H}}\rangle^{-\mathbf{1}}_{(3.59)}e^{-v_1\,\hat{T}^{-1}\,\langle\hat{\mathcal{H}}\rangle_{(3.59)}\,\hat{T}\,\langle\hat{\mathcal{H}}\rangle^{-\mathbf{1}}_{(3.59)}}\Bigg)+\Big(\text{sub-indices }\;1\leftrightarrow 2\Big)\Bigg)\times\\
\times\;& \left.\Bigg(\int_0^{+\infty}du\;\hat{1}\;e^{-u\,(v_1+v_2+v_3)}+\imath\Big(\hat{T}^{-1}\,\hat{I}\,\hat{\eta}\Big|\widehat{J_\psi}\Big\rangle\otimes\Big\langle\widehat{J_\psi}\Big|\hat{\eta}\,\hat{I}\hat{T}\,\Big\langle\hat{\mathcal{H}}\Big\rangle^{-\mathbf{1}}_{(3.59)}\Big)\Bigg)\Bigg]\right\}\;.
\end{aligned}
$$

4.3 Gauge transformation to the interaction representation of pure gradient terms

The gradient term $\Delta\hat{\mathcal{H}}^{ba}_{N;M}(y_q,x_p)$ (4.5,4.24) depends on the detailed mean field potential $\langle\hat{\mathcal{V}}(x_p)\rangle_{(3.59)}$ as the appropriate interaction; however, we can perform a gradient expansion of $\mathcal{A}_{DET}[\hat{T},\langle\hat{\mathcal{V}}\rangle_{(3.59)};\hat{\partial}]$ (4.2), $\mathcal{A}_{J_\psi}[\hat{T},\langle\hat{\mathcal{V}}\rangle;\hat{\partial}]$ (4.3) with '*universal properties*' by changing to the 'interaction representation' (4.48,4.49) of the coset matrices. We assume general, complex- and even-valued, block diagonal matrices $\hat{\mathfrak{W}}^{bb}_{N_0\times N_0}(x_p)$, $\hat{\mathfrak{W}}^{-1;aa}_{N_0\times N_0}(x_p)$ for this transformation so that the matrix $\hat{T}^{ab}_{\hat{\mathfrak{W}}}(x_p)$ (4.48) follows from $\hat{T}^{ab}(x_p)$ in the 'interaction representation' by choosing a suitable dependence of $\hat{\mathfrak{W}}^{bb}_{N_0\times N_0}(x_p)$, $\hat{\mathfrak{W}}^{-1;aa}_{N_0\times N_0}(x_p)$ on the potential $\langle\hat{\mathcal{V}}(x_p)\rangle_{(3.59)}$. Since the '22' part '$\hat{\mathcal{H}}^{22}_{\hat{\mathfrak{W}}}(x_p)$' is given as the transpose of the '11' part '$(\hat{\mathcal{H}}^{11}_{\hat{\mathfrak{W}}}(x_p))^T$' (4.50,4.51), one has to require relations (4.52-4.54) where we introduce a general, complex- and even-valued generator $\hat{\mathfrak{w}}_{N_0\times N_0}(x_p)$ (4.54) for the block diagonal transformation with $\hat{\mathfrak{W}}^{bb}_{N_0\times N_0}(x_p)$, $\hat{\mathfrak{W}}^{-1;aa}_{N_0\times N_0}(x_p)$

$$
\begin{aligned}
\hat{T}^{ab}(x_p) &\;\rightarrow\; \hat{T}^{ab}_{\hat{\mathfrak{W}}}(x_p)=\hat{\mathfrak{W}}^{-1;aa}(x_p)\,\hat{T}^{ab}(x_p)\,\hat{\mathfrak{W}}^{bb}(x_p)\;; && (4.48)\\
\hat{T}^{-1;ab}(x_p) &\;\rightarrow\; \hat{T}^{-1;ab}_{\hat{\mathfrak{W}}}(x_p)=\hat{\mathfrak{W}}^{-1;aa}(x_p)\,\hat{T}^{-1;ab}(x_p)\,\hat{\mathfrak{W}}^{bb}(x_p)\;; && (4.49)\\
\hat{\mathcal{H}}^{a=b}(x_p) &\;\rightarrow\; \hat{\mathcal{H}}^{a=b}_{\hat{\mathfrak{W}}}(x_p)=\hat{\mathfrak{W}}^{-1;aa}(x_p)\,\hat{\mathcal{H}}^{a=b}(x_p)\,\hat{\mathfrak{W}}^{bb}(x_p)\;; && (4.50)\\
\Big(\hat{\mathcal{H}}^{11}_{\hat{\mathfrak{W}}}(x_p)\Big)^T &\;=\; \hat{\mathcal{H}}^{22}_{\hat{\mathfrak{W}}}(x_p)\Longrightarrow\hat{\mathfrak{W}}^{22}(x_p)=\Big(\hat{\mathfrak{W}}^{-1;11}(x_p)\Big)^T\;; && (4.51)\\
\hat{\mathfrak{W}}^{11}(x_p) &\;=\; \exp\big\{\hat{\mathfrak{w}}(x_p)\big\}\;;\quad \hat{\mathfrak{W}}^{11;-1}(x_p)=\exp\big\{-\hat{\mathfrak{w}}(x_p)\big\}\;; && (4.52)\\
\hat{\mathfrak{W}}^{22}(x_p) &\;=\; \exp\big\{-\hat{\mathfrak{w}}^T(x_p)\big\}\;;\quad \hat{\mathfrak{W}}^{22;-1}(x_p)=\exp\big\{\hat{\mathfrak{w}}^T(x_p)\big\}\;; && (4.53)\\
\hat{\mathfrak{w}}(x_p) &\;:=\; \hat{\mathfrak{w}}_{N_0\times N_0}(x_p)\in\mathsf{C}_{\mathsf{even}}\;. && (4.54)
\end{aligned}
$$

The suitable choice of gauge (4.55) for $\exp\{\hat{\mathfrak{w}}(x_p)\}$ is determined in such a dependence on the potential $\langle\hat{\mathcal{V}}(x_p)\rangle_{(3.59)}$ and mass term that the interaction representation $\hat{\mathcal{H}}^{aa}_{\hat{\mathfrak{W}}}(x_p)$ reduces to the contour spacetime gradients $\hat{\boldsymbol{\partial}}_{\boldsymbol{p},\boldsymbol{\mu}}$ which are dressed by the matrices $\exp\{\pm\hat{\mathfrak{w}}_{N_0\times N_0}(x_p)\}$, $\exp\{\pm\hat{\mathfrak{w}}^T_{N_0\times N_0}(x_p)\}$ aside from the gamma matrices $\hat{\beta}$, $\hat{\gamma}^\mu$. (Note that one has to distinguish between saturated derivatives, as e.g. $(\hat{\partial}_{p,\mu}\hat{T}(x_p))$, and unsaturated gradient operators $\hat{\boldsymbol{\partial}}_{\boldsymbol{p},\boldsymbol{\mu}}$ (typed in boldface) acting further to the right or left beyond the coset matrices!). Therefore, one finds for the interaction representation of $\hat{\mathcal{H}}^{aa}_{\hat{\mathfrak{W}}}(x_p)$, in which the potential $\langle\hat{\mathcal{V}}(x_p)\rangle_{(3.59)}$ and mass $\hat{m}$ are removed by the chosen gauge (4.55), the simplified relations (4.56,4.57)

$$
\begin{aligned}
-\Big(\imath\,\langle\hat{\mathcal{V}}(x_p)\rangle+\hat{m}\Big) &= \Big(\hat{\partial\!\!\!/}_p\exp\big\{\hat{\mathfrak{w}}(x_p)\big\}\Big)\;\exp\big\{-\hat{\mathfrak{w}}(x_p)\big\}\;; && (4.55)\\
\hat{\mathcal{H}}^{11}_{\hat{\mathfrak{W}}}(x_p) &= \exp\big\{-\hat{\mathfrak{w}}(x_p)\big\}\;\hat{H}_p(x_p)\;\exp\big\{\hat{\mathfrak{w}}(x_p)\big\} && (4.56)\\
&= \exp\big\{-\hat{\mathfrak{w}}(x_p)\big\}\;\hat{\beta}\,\hat{\gamma}^\mu\;\exp\big\{\hat{\mathfrak{w}}(x_p)\big\}\;\hat{\boldsymbol{\partial}}_{\boldsymbol{p},\boldsymbol{\mu}}-\imath\,\varepsilon_p\;\hat{1}_{N_0\times N_0}\;;\\
\hat{\mathcal{H}}^{22}_{\hat{\mathfrak{W}}}(x_p) &= \Big(\hat{\mathcal{H}}^{11}_{\hat{\mathfrak{W}}}(x_p)\Big)^T=\exp\big\{\hat{\mathfrak{w}}^T(x_p)\big\}\;\hat{H}^T_p(x_p)\;\exp\big\{-\hat{\mathfrak{w}}^T(x_p)\big\} && (4.57)\\
&= -\hat{\boldsymbol{\partial}}_{\boldsymbol{p},\boldsymbol{\mu}}\exp\big\{\hat{\mathfrak{w}}^T(x_p)\big\}\;(\hat{\beta}\hat{\gamma}^\mu)^T\;\exp\big\{-\hat{\mathfrak{w}}^T(x_p)\big\}-\imath\,\varepsilon_p\;\hat{1}_{N_0\times N_0}\;.
\end{aligned}
$$

The successive definitions and steps (4.58-4.62) for the interaction picture of Dirac gamma matrices lead to anomalous doubled, block diagonal matrices $\hat{\mathcal{B}}_{\hat{\mathfrak{W}}}^{aa}(x_p)$, $\hat{\Gamma}_{\hat{\mathfrak{W}}}^{\mu,aa}(x_p)$ which fulfill the identical Clifford algebra for 3+1 spacetime dimensions despite of their local spacetime dependence

$$\hat{\mathcal{B}}_{\hat{\mathfrak{W}}}^{11}(x_p) = e^{-\hat{\mathfrak{w}}(x_p)}\;\hat{\beta}\;e^{\hat{\mathfrak{w}}(x_p)}\;; \tag{4.58}$$

$$\hat{\mathcal{B}}_{\hat{\mathfrak{W}}}^{22}(x_p) = \Big(\hat{\mathcal{B}}_{\hat{\mathfrak{W}}}^{11}(x_p)\Big)^T = e^{\hat{\mathfrak{w}}^T(x_p)}\;\hat{\beta}\;e^{-\hat{\mathfrak{w}}^T(x_p)}\;;\quad \hat{\beta}^T=\hat{\beta}\;; \tag{4.59}$$

$$\hat{\mathcal{B}}_{\hat{\mathfrak{W}}}(x_p)\,\hat{\Gamma}_{\hat{\mathfrak{W}}}^{\mu}(x_p) = \begin{pmatrix} e^{-\hat{\mathfrak{w}}(x_p)} & \\ & e^{\hat{\mathfrak{w}}^T(x_p)}\end{pmatrix}\begin{pmatrix}\hat{\beta}\,\hat{\gamma}^\mu & \\ & \big(\hat{\beta}\,\hat{\gamma}^\mu\big)^T\end{pmatrix}\begin{pmatrix} e^{\hat{\mathfrak{w}}(x_p)} & \\ & e^{-\hat{\mathfrak{w}}^T(x_p)}\end{pmatrix}; \tag{4.60}$$

$$\hat{\mathcal{B}}_{\hat{\mathfrak{W}}}^{11}(x_p)\,\hat{\Gamma}_{\hat{\mathfrak{W}}}^{\mu,11}(x_p) = e^{-\hat{\mathfrak{w}}(x_p)}\;\hat{\beta}\;e^{\hat{\mathfrak{w}}(x_p)}\;\;e^{-\hat{\mathfrak{w}}(x_p)}\;\hat{\gamma}^\mu\;e^{\hat{\mathfrak{w}}(x_p)}\;; \tag{4.61}$$

$$\hat{\mathcal{B}}_{\hat{\mathfrak{W}}}^{22}(x_p)\,\hat{\Gamma}_{\hat{\mathfrak{W}}}^{\mu,22}(x_p) = \Big(\hat{\mathcal{B}}_{\hat{\mathfrak{W}}}^{11}(x_p)\,\hat{\Gamma}_{\hat{\mathfrak{W}}}^{\mu,11}(x_p)\Big)^T = e^{\hat{\mathfrak{w}}^T(x_p)}\;\big(\hat{\beta}\,\hat{\gamma}^\mu\big)^T\;e^{-\hat{\mathfrak{w}}^T(x_p)}\;;\quad \big(\hat{\gamma}^0\big)^T=\hat{\gamma}^0\,. \tag{4.62}$$

Application of (4.58-4.62) with the chosen gauge (4.55) reduces the anomalous doubled, one-particle Hamiltonian to pure gradient terms (4.63) with locally transformed gamma matrices in the interaction picture for $\hat{T}_{\hat{\mathfrak{W}}}^{ab}(x_p)$. The anomalous doubled, one-particle Hamiltonian or contour spacetime gradient $\hat{\mathcal{H}}_{\hat{\mathfrak{W}}}(x_p)$ (4.63) is further decomposed into commutator and anti-commutator parts (4.64-4.66) with spacetime dependent gamma matrices; in consequence one eventually attains the Hamiltonian $\hat{\mathcal{H}}_{\hat{\mathfrak{W}}}(x_p)$ (4.67) of the interaction representation with unsaturated gradients '$\boldsymbol{\hat{\partial}_{p,\mu}}$' and saturated derivatives $(\hat{\partial}_{p,\mu}\hat{\mathcal{B}}_{\hat{\mathfrak{W}}}^{aa}(x_p)\,\hat{\Gamma}_{\hat{\mathfrak{W}}}^{\mu,aa}(x_p)\,)$ of gamma matrices

$$\hat{\mathcal{H}}_{\hat{\mathfrak{W}}}(x_p) = \begin{pmatrix}\hat{\mathcal{B}}_{\hat{\mathfrak{W}}}^{11}(x_p)\,\hat{\Gamma}_{\hat{\mathfrak{W}}}^{\mu,11}(x_p)\,\boldsymbol{\hat{\partial}_{p,\mu}} - \imath\,\varepsilon_p\,\hat{1}_{N_0\times N_0} & 0 \\ 0 & -\boldsymbol{\hat{\partial}_{p,\mu}}\,\hat{\mathcal{B}}_{\hat{\mathfrak{W}}}^{22}(x_p)\,\hat{\Gamma}_{\hat{\mathfrak{W}}}^{\mu,22}(x_p) - \imath\,\varepsilon_p\,\hat{1}_{N_0\times N_0}\end{pmatrix}; \tag{4.63}$$

$$\begin{aligned}\hat{\mathcal{H}}_{\hat{\mathfrak{W}}}(x_p) &= -\imath\,\varepsilon_p\,\hat{1}_{2N_0\times 2N_0} + \\ &+\frac{1}{2}\begin{pmatrix}\big[\hat{\mathcal{B}}_{\hat{\mathfrak{W}}}^{11}(x_p)\,\hat{\Gamma}_{\hat{\mathfrak{W}}}^{\mu,11}(x_p)\,,\,\boldsymbol{\hat{\partial}_{p,\mu}}\big]_- & \\ & \big[\hat{\mathcal{B}}_{\hat{\mathfrak{W}}}^{22}(x_p)\,\hat{\Gamma}_{\hat{\mathfrak{W}}}^{\mu,22}(x_p)\,,\,\boldsymbol{\hat{\partial}_{p,\mu}}\big]_-\end{pmatrix} + \\ &+\frac{1}{2}\begin{pmatrix}\big\{\hat{\mathcal{B}}_{\hat{\mathfrak{W}}}^{11}(x_p)\,\hat{\Gamma}_{\hat{\mathfrak{W}}}^{\mu,11}(x_p)\,,\,\boldsymbol{\hat{\partial}_{p,\mu}}\big\}_+ & \\ & -\big\{\hat{\mathcal{B}}_{\hat{\mathfrak{W}}}^{22}(x_p)\,\hat{\Gamma}_{\hat{\mathfrak{W}}}^{\mu,22}(x_p)\,,\,\boldsymbol{\hat{\partial}_{p,\mu}}\big\}_+\end{pmatrix};\end{aligned} \tag{4.64}$$

$$\big[\hat{\mathcal{B}}_{\hat{\mathfrak{W}}}^{aa}(x_p)\,\hat{\Gamma}_{\hat{\mathfrak{W}}}^{\mu,aa}(x_p)\,,\,\boldsymbol{\hat{\partial}_{p,\mu}}\big]_- = -\big(\hat{\partial}_{p,\mu}\hat{\mathcal{B}}_{\hat{\mathfrak{W}}}^{aa}(x_p)\,\hat{\Gamma}_{\hat{\mathfrak{W}}}^{\mu,aa}(x_p)\big)\;; \tag{4.65}$$

$$\big\{\hat{\mathcal{B}}_{\hat{\mathfrak{W}}}^{aa}(x_p)\,\hat{\Gamma}_{\hat{\mathfrak{W}}}^{\mu,aa}(x_p)\,,\,\boldsymbol{\hat{\partial}_{p,\mu}}\big\}_+ = 2\,\hat{\mathcal{B}}_{\hat{\mathfrak{W}}}^{aa}(x_p)\,\hat{\Gamma}_{\hat{\mathfrak{W}}}^{\mu,aa}(x_p)\,\boldsymbol{\hat{\partial}_{p,\mu}} + \big(\hat{\partial}_{p,\mu}\hat{\mathcal{B}}_{\hat{\mathfrak{W}}}^{aa}(x_p)\,\hat{\Gamma}_{\hat{\mathfrak{W}}}^{\mu,aa}(x_p)\big)\;; \tag{4.66}$$

$$\begin{aligned}\hat{\mathcal{H}}_{\hat{\mathfrak{W}}}(x_p) &= -\imath\,\varepsilon_p\,\hat{1}_{2N_0\times 2N_0} - \frac{1}{2}\,\delta_{ab}\,\big(\hat{\partial}_{p,\mu}\hat{\mathcal{B}}_{\hat{\mathfrak{W}}}^{aa}(x_p)\,\hat{\Gamma}_{\hat{\mathfrak{W}}}^{\mu,aa}(x_p)\big) + \frac{1}{2}\,\hat{S}\,\delta_{ab}\,\big(\hat{\partial}_{p,\mu}\hat{\mathcal{B}}_{\hat{\mathfrak{W}}}^{aa}(x_p)\,\hat{\Gamma}_{\hat{\mathfrak{W}}}^{\mu,aa}(x_p)\big) + \\ &+\delta_{ab}\,\hat{S}\,\hat{\mathcal{B}}_{\hat{\mathfrak{W}}}^{aa}(x_p)\,\hat{\Gamma}_{\hat{\mathfrak{W}}}^{\mu,aa}(x_p)\,\boldsymbol{\hat{\partial}_{p,\mu}}\,.\end{aligned} \tag{4.67}$$

The change to the interaction picture transforms the generating function (3.110-3.116) with $\hat{T}(x_p)$ to the corresponding path integral $Z[\hat{\mathcal{J}}_{\hat{\mathfrak{W}}},J_{\psi;\hat{\mathfrak{W}}},\hat{J}_{\psi\psi}]$ (4.68) with colour-dressed source terms $J_{\psi;\hat{\mathfrak{W}}}$, $\hat{\mathcal{J}}_{\hat{\mathfrak{W}}}$ and coset matrices $\hat{T}_{\hat{\mathfrak{W}}}(x_p)$

$$\begin{aligned}Z[\hat{\mathcal{J}}_{\hat{\mathfrak{W}}},J_{\psi;\hat{\mathfrak{W}}},\hat{J}_{\psi\psi}] &= \int d[\hat{T}^{-1}(x_p)\,d\hat{T}(x_p)]\;\;\big\langle Z_{\hat{J}_{\psi\psi}}[\hat{\mathfrak{W}}\,\hat{T}_{\hat{\mathfrak{W}}}\,\hat{\mathfrak{W}}^{-1}]\big\rangle\times \\ &\times\;\;\exp\Big\{\mathcal{A}_{DET}[\hat{T}_{\hat{\mathfrak{W}}};\hat{\mathcal{J}}_{\hat{\mathfrak{W}}}] + \imath\,\mathcal{A}_{J_{\psi;\hat{\mathfrak{W}}}}[\hat{T}_{\hat{\mathfrak{W}}};\hat{\mathcal{J}}_{\hat{\mathfrak{W}}}]\Big\}\;;\end{aligned} \tag{4.68}$$

$$\mathcal{A}_{DET}[\hat{T}_{\hat{\mathfrak{W}}};\hat{\mathcal{J}}_{\hat{\mathfrak{W}}}] = \frac{1}{2}\int_C d^4x_p\;\eta_p\;\mathcal{N}\underset{N_f,\hat{\gamma}_{mn}^{(\mu)},N_c}{\overset{a(=1,2)}{\mathfrak{Tr}}}\bigg(\ln\Big[\hat{\mathcal{O}}_{\hat{\mathfrak{W}};N;M}^{ba}(y_q,x_p)\Big] - \ln\Big[\hat{\mathcal{H}}_{\hat{\mathfrak{W}};N;M}^{ba}(y_q,x_p)\Big]\bigg)\;; \tag{4.69}$$

$$\begin{aligned}
\mathcal{A}_{J_{\psi;\hat{\mathfrak{W}}}}[\hat{T}_{\hat{\mathfrak{W}}};\hat{\mathfrak{J}}_{\hat{\mathfrak{W}}}] &= \frac{1}{2}\int_C d^4x_p\; d^4y_q\Big(J^{\dagger,b}_{\psi;N_1}(y_q)\;\hat{\mathfrak{W}}^{bb}_{N_1;N}(y_q)\Big)\;\hat{I}\;\times \\
&\times\;\Big(\hat{T}^{bb'}_{\hat{\mathfrak{W}};N;N'}(y_q)\;\hat{\mathfrak{O}}^{-\mathbf{1};b'a'}_{\hat{\mathfrak{W}};N';M'}(y_q,x_p)\;\hat{T}^{-1;a'a}_{\hat{\mathfrak{W}};M';M}(x_p)-\hat{\mathcal{H}}^{-\mathbf{1};ba}_{\hat{\mathfrak{W}};N;M}(y_q,x_p)\Big)\;\times \\
&\times\;\hat{I}\;\Big(\hat{\mathfrak{W}}^{-1;aa}_{M;M_1}(x_p)\;J^{a}_{\psi;M_1}(x_p)\Big)\;;
\end{aligned} \tag{4.70}$$

$$\begin{aligned}
\hat{\mathfrak{O}}^{ba}_{\hat{\mathfrak{W}};N;M}(y_q,x_p) &= \Big\{\hat{\mathcal{H}}_{\hat{\mathfrak{W}}}+\Big(\hat{T}^{-1}_{\hat{\mathfrak{W}}}\;\hat{\mathcal{H}}_{\hat{\mathfrak{W}}}\;\hat{T}_{\hat{\mathfrak{W}}}-\hat{\mathcal{H}}_{\hat{\mathfrak{W}}}\Big)+ \\
&+\;\hat{T}^{-1}_{\hat{\mathfrak{W}}}\;\hat{I}\;\hat{S}\;\eta_q\;\hat{\mathfrak{W}}^{-1;b'b'}_{N';N_1}(y_q)\frac{\hat{\mathfrak{J}}^{b'a'}_{N_1;M_1}(y_q,x_p)}{\mathcal{N}}\hat{\mathfrak{W}}^{a'a'}_{M_1;M'}(x_p)\;\eta_p\;\hat{S}\;\hat{I}\;\hat{T}_{\hat{\mathfrak{W}}}\Big\}^{ba}_{N;M}(y_q,x_p)\;.
\end{aligned} \tag{4.71}$$

It has to be pointed out that the transformation to the interaction picture cannot be incorporated into the coset integration measure $d[\hat{T}^{-1}(x_p)\,d\hat{T}(x_p)]\not\Rightarrow d[\hat{T}^{-1}_{\hat{\mathfrak{W}}}(x_p)\,d\hat{T}_{\hat{\mathfrak{W}}}(x_p)]$ because the generator, determined by the saddle point approximation of $\langle\hat{\mathcal{V}}(x_p)\rangle_{(3.59)}$, consists of a *general*, complex-valued matrix $\hat{\mathfrak{w}}_{N_0\times N_0}(x_p)$ instead of the necessarily hermitian generator $\hat{\mathcal{G}}_{D;N_0\times N_0}(x_p)$ for the diagonalizing matrices $\hat{P}^{aa}_{N_0\times N_0}(x_p)$; the hermitian-conjugation symmetry between $\hat{X}(x_p)$, $\hat{X}^{\dagger}(x_p)$ as sub-generators of $\hat{Y}(x_p)$ does not persist in the interaction picture with $\hat{Y}_{\hat{\mathfrak{W}}}(x_p)=\hat{\mathfrak{W}}^{-1}(x_p)\;\hat{Y}(x_p)\;\hat{\mathfrak{W}}(x_p)$ because of $(\hat{X}_{\hat{\mathfrak{W}}}(x_p)\,)^{\dagger}\neq(\hat{X}^{\dagger}_{\hat{\mathfrak{W}}}(x_p)\,)$ due to the completely arbitray complex matrix structure of $\hat{\mathfrak{w}}_{N_0\times N_0}(x_p)$ (only restricted by the '$-\imath\,\hat{\varepsilon}_p$' term). As we specify the operators in $\mathcal{A}_{DET}[\hat{T}_{\hat{\mathfrak{W}}};\hat{\mathfrak{J}}_{\hat{\mathfrak{W}}}\equiv 0]$, $\mathcal{A}_{J_{\psi;\hat{\mathfrak{W}}}}[\hat{T}_{\hat{\mathfrak{W}}};\hat{\mathfrak{J}}_{\hat{\mathfrak{W}}}\equiv 0]$ in correspondence to section 4.2, we obtain the combination $\hat{T}^{-1}_{\hat{\mathfrak{W}}}\,\hat{\mathcal{H}}_{\hat{\mathfrak{W}}}\,\hat{T}_{\hat{\mathfrak{W}}}$ of pure gradients terms and also the inverse weight $\hat{T}_{\hat{\mathfrak{W}}}\,\hat{\mathcal{H}}^{-\mathbf{1}}_{\hat{\mathfrak{W}}}\,\hat{T}^{-1}_{\hat{\mathfrak{W}}}$ due to the trace operations. Thus, the transformation to the interaction picture points out the problem of finite order gradients, having neither a small nor large momentum expansion

$$\begin{aligned}
\mathcal{A}_{DET}[\hat{T}_{\hat{\mathfrak{W}}};\hat{\mathfrak{J}}_{\hat{\mathfrak{W}}}\equiv 0] &= \frac{1}{2}\;\underset{\int_C d^4x_p\;\eta_p}{\overset{a(=1,2)}{\mathrm{TR}}}\;\underset{N_f,\hat{\gamma}^{(\mu)}_{mn},N_c}{\mathfrak{tr}}\Big[\ln\Big[\hat{T}^{-1}_{\hat{\mathfrak{W}}}\;\hat{\mathcal{H}}_{\hat{\mathfrak{W}}}\;\hat{T}_{\hat{\mathfrak{W}}}\;\hat{\mathcal{H}}^{-\mathbf{1}}_{\hat{\mathfrak{W}}}\Big]\Big] \\
&= \frac{1}{2}\int_0^{+\infty}dv\;\underset{\int_C d^4x_p\;\eta_p}{\overset{a(=1,2)}{\mathrm{TR}}}\;\underset{N_f,\hat{\gamma}^{(\mu)}_{mn},N_c}{\mathfrak{tr}}\left[\frac{\exp\{-v\;\hat{1}\}-\exp\Big\{-v\,\hat{T}^{-1}_{\hat{\mathfrak{W}}}\;\hat{\mathcal{H}}_{\hat{\mathfrak{W}}}\;\hat{T}_{\hat{\mathfrak{W}}}\;\hat{\mathcal{H}}^{-\mathbf{1}}_{\hat{\mathfrak{W}}}\Big\}}{v}\right];
\end{aligned} \tag{4.72}$$

$$\begin{aligned}
\mathcal{A}_{J_{\psi;\hat{\mathfrak{W}}}}[\hat{T}_{\hat{\mathfrak{W}}};\hat{\mathfrak{J}}_{\hat{\mathfrak{W}}}\equiv 0] &= \frac{\imath}{2}\langle\widehat{J_{\psi;\hat{\mathfrak{W}}}}|\hat{\eta}\;\hat{I}\;\Big(\hat{T}_{\hat{\mathfrak{W}}}\;\hat{\mathcal{H}}^{-\mathbf{1}}_{\hat{\mathfrak{W}}}\;\Big[\hat{1}+\Delta\hat{\mathcal{H}}_{\hat{\mathfrak{W}}}\;\hat{\mathcal{H}}^{-\mathbf{1}}_{\hat{\mathfrak{W}}}\Big]^{-\mathbf{1}}\hat{T}^{-1}_{\hat{\mathfrak{W}}}-\hat{\mathcal{H}}^{-\mathbf{1}}_{\hat{\mathfrak{W}}}\Big)\;\hat{I}\;\hat{\eta}|\widehat{J_{\psi;\hat{\mathfrak{W}}}}\rangle \\
&= \frac{\imath}{2}\langle\widehat{J_{\psi;\hat{\mathfrak{W}}}}|\hat{\eta}\;\hat{I}\;\Big(\int_0^{+\infty}dv\;\;\hat{T}_{\hat{\mathfrak{W}}}\;\hat{\mathcal{H}}^{-\mathbf{1}}_{\hat{\mathfrak{W}}}\;\exp\Big\{-v\,\hat{T}^{-1}_{\hat{\mathfrak{W}}}\;\hat{\mathcal{H}}_{\hat{\mathfrak{W}}}\;\hat{T}_{\hat{\mathfrak{W}}}\;\hat{\mathcal{H}}^{-\mathbf{1}}_{\hat{\mathfrak{W}}}\Big\}\;\hat{T}^{-1}_{\hat{\mathfrak{W}}}-\hat{\mathcal{H}}^{-\mathbf{1}}_{\hat{\mathfrak{W}}}\Big)\;\hat{I}\;\hat{\eta}|\widehat{J_{\psi;\hat{\mathfrak{W}}}}\rangle\;.
\end{aligned} \tag{4.73}$$

4.4 Green functions of gauge transformed gradient terms in gradually varying background fields

The change to the interaction picture transforms the generating function (3.110-3.116) with $\hat{T}(x_p)$ to the corresponding path integral $Z[\hat{\mathfrak{J}}_{\hat{\mathfrak{W}}},J_{\psi;\hat{\mathfrak{W}}},\hat{J}_{\psi\psi}]$ (4.74) with colour-dressed source terms $J_{\psi;\hat{\mathfrak{W}}}$, $\hat{\mathfrak{J}}_{\hat{\mathfrak{W}}}$ and coset matrices $\hat{T}_{\hat{\mathfrak{W}}}(x_p)$

$$\begin{aligned}
Z[\hat{\mathfrak{J}}_{\hat{\mathfrak{W}}},J_{\psi;\hat{\mathfrak{W}}},\hat{J}_{\psi\psi}] &= \int d[\hat{T}^{-1}(x_p)\;d\hat{T}(x_p)]\;\;Z_{\hat{J}_{\psi\psi}}[\hat{\mathfrak{W}}\,\hat{T}_{\hat{\mathfrak{W}}}\,\hat{\mathfrak{W}}^{-1}]\;\times \\
&\times\;\exp\Big\{\mathcal{A}_{DET}[\hat{T}_{\hat{\mathfrak{W}}};\hat{\mathfrak{J}}_{\hat{\mathfrak{W}}}]+\imath\,\mathcal{A}_{J_{\psi;\hat{\mathfrak{W}}}}[\hat{T}_{\hat{\mathfrak{W}}};\hat{\mathfrak{J}}_{\hat{\mathfrak{W}}}]\Big\}\;;
\end{aligned} \tag{4.74}$$

$$\mathcal{A}_{DET}[\hat{T}_{\hat{\mathfrak{W}}};\hat{\mathfrak{J}}_{\hat{\mathfrak{W}}}] = \frac{1}{2}\int_C d^4x_p\;\eta_p\;\mathcal{N}\underset{N_f,\hat{\gamma}^{(\mu)}_{mn},N_c}{\overset{a(=1,2)}{\mathfrak{Tr}}}\;\ln\Big[\hat{\mathfrak{O}}^{ba}_{\hat{\mathfrak{W}};N;M}(y_q,x_p)\Big]\;; \tag{4.75}$$

$$\mathcal{A}_{J_{\psi;\hat{\mathfrak{W}}}}[\hat{T}_{\hat{\mathfrak{W}}};\hat{\mathfrak{J}}_{\hat{\mathfrak{W}}}] = \frac{1}{2}\int_C d^4x_p\;d^4y_q\Big(J^{\dagger,b}_{\psi;N_1}(y_q)\;\hat{\mathfrak{W}}^{bb}_{N_1;N}(y_q)\Big)\;\times \tag{4.76}$$

$$\begin{aligned}
&\times\quad \hat{I}\;\hat{T}^{bb'}_{\hat{\mathfrak{W}};N;N'}(y_q)\;\hat{\mathcal{O}}^{-1;b'a'}_{\hat{\mathfrak{W}};N';M'}(y_q,x_p)\;\hat{T}^{-1;a'a}_{\hat{\mathfrak{W}};M';M}(x_p)\;\hat{I}\;\times\;\Big(\hat{\mathfrak{W}}^{-1;aa}_{M;M_1}(x_p)\;J^{a}_{\psi;M_1}(x_p)\Big)\;; \\
\hat{\mathcal{O}}^{ba}_{\hat{\mathfrak{W}};N;M}(y_q,x_p) &= \Big\{\hat{\mathcal{H}}_{\hat{\mathfrak{W}}}+\Big(\hat{T}^{-1}_{\hat{\mathfrak{W}}}\;\hat{\mathcal{H}}_{\hat{\mathfrak{W}}}\;\hat{T}_{\hat{\mathfrak{W}}}-\hat{\mathcal{H}}_{\hat{\mathfrak{W}}}\Big)+ \\
&+\;\hat{T}^{-1}_{\hat{\mathfrak{W}}}\;\hat{I}\;\hat{S}\;\eta_q\;\hat{\mathfrak{W}}^{-1;b'b'}_{N';N_1}(y_q)\frac{\hat{\mathfrak{J}}^{b'a'}_{N_1;M_1}(y_q,x_p)}{\mathcal{N}}\hat{\mathfrak{W}}^{a'a'}_{M_1;M'}(x_p)\;\eta_p\;\hat{S}\;\hat{I}\;\hat{T}_{\hat{\mathfrak{W}}}\Big\}^{ba}_{N;M}(y_q,x_p)\;.
\end{aligned} \tag{4.77}$$

According to Derrick's theorem [13], a consistent gradient expansion requires terms up to the order of four for stable, static energy configurations and also the assumption of slowly varying spacetime dependent background fields as $\langle\hat{\mathcal{V}}(x_p)\rangle$. We therefore neglect the 'saturated' derivatives (4.78) of slowly varying gamma matrices in $\hat{\mathcal{H}}_{\hat{\mathfrak{W}}}(x_p)$ (4.63,4.64) so that the approximated gradient term $\Delta\hat{\mathcal{H}}^{ba}_{\hat{\mathfrak{W}};N;M}(y_q,x_p)$ (4.79) follows in place of the original defined one

$$0 \;\approx\; \big(\hat{\partial}_{p,\mu}\hat{\beta}^{aa}_{\hat{\mathfrak{W}}}(x_p)\;\hat{\gamma}^{\mu,aa}_{\hat{\mathfrak{W}}}(x_p)\big)\;; \tag{4.78}$$

$$\begin{aligned}
&\Delta\hat{\mathcal{H}}^{ba}_{\hat{\mathfrak{W}};N;M}(y_q,x_p)=\Big(\hat{T}^{-1}_{\hat{\mathfrak{W}}}\hat{\mathcal{H}}_{\hat{\mathfrak{W}}}\hat{T}_{\hat{\mathfrak{W}}}-\hat{\mathcal{H}}_{\hat{\mathfrak{W}}}\Big)^{ba}_{N;M}(y_q,x_p)\approx\delta^{(4)}(y_q-x_p)\;\eta_q\;\times \\
&\times\Big[\hat{T}^{-1;bb'}_{\hat{\mathfrak{W}};N;N'}(x_p)\;\hat{S}\;\delta_{b'a'}\Big(\hat{\beta}^{a'a'}_{\hat{\mathfrak{W}}}(x_p)\;\hat{\gamma}^{\mu,a'a'}_{\hat{\mathfrak{W}}}(x_p)\Big)^{a'a'}_{N';M'}\;\Big(\hat{\partial}_{p,\mu}\hat{T}^{a'a}_{\hat{\mathfrak{W}};M';M}(x_p)\Big)+ \\
&+\Big\{\hat{T}^{-1;bb'}_{\hat{\mathfrak{W}};N;N'}(x_p)\;\hat{S}\;\delta_{b'a'}\Big(\hat{\beta}^{a'a'}_{\hat{\mathfrak{W}}}(x_p)\;\hat{\gamma}^{\mu,a'a'}_{\hat{\mathfrak{W}}}(x_p)\Big)^{a'a'}_{N';M'}\;\hat{T}^{a'a}_{\hat{\mathfrak{W}};M';M}(x_p)-\hat{S}\;\delta_{ba}\Big(\hat{\beta}^{aa}_{\hat{\mathfrak{W}}}(x_p)\;\hat{\gamma}^{\mu,aa}_{\hat{\mathfrak{W}}}(x_p)\Big)^{ba}_{N;M}\Big\}\boldsymbol{\hat{\partial}_{p,\mu}}\Big].
\end{aligned} \tag{4.79}$$

The completely labeled gradient term $\Delta\hat{\mathcal{H}}^{ba}_{\hat{\mathfrak{W}};N;M}(y_q,x_p)$ (4.79) is abbreviated by Eq. (4.80) with assumed constant, block diagonal gamma matrices $\hat{\beta}^{aa}_{\hat{\mathfrak{W}}}$, $\hat{\gamma}^{\mu,aa}_{\hat{\mathfrak{W}}}$

$$\begin{aligned}
\Delta\hat{\mathcal{H}}_{\hat{\mathfrak{W}}}(y_q,x_p) &\approx \delta^{(4)}(y_q-x_p)\;\eta_q\;\Big[\hat{T}^{-1}_{\hat{\mathfrak{W}}}(x_p)\;\hat{S}\;\hat{\beta}^{aa}_{\hat{\mathfrak{W}}}\;\hat{\gamma}^{\mu,aa}_{\hat{\mathfrak{W}}}\;\Big(\hat{\partial}_{p,\mu}\hat{T}_{\hat{\mathfrak{W}}}(x_p)\Big)+ \\
&+\;\Big\{\hat{T}^{-1}_{\hat{\mathfrak{W}}}(x_p)\;\hat{S}\;\hat{\beta}^{aa}_{\hat{\mathfrak{W}}}\;\hat{\gamma}^{\mu,aa}_{\hat{\mathfrak{W}}}\;\hat{T}_{\hat{\mathfrak{W}}}(x_p)-\hat{S}\;\hat{\beta}^{aa}_{\hat{\mathfrak{W}}}\;\hat{\gamma}^{\mu,aa}_{\hat{\mathfrak{W}}}\Big\}\boldsymbol{\hat{\partial}_{p,\mu}}\Big]\;.
\end{aligned} \tag{4.80}$$

Proceeding as in chapter 4 of [11], one has to determine the anomalous-doubled time contour Green function $\hat{G}^{(0)}=[\hat{\mathcal{H}}_{\hat{\mathfrak{W}}}(x_p)]^{-1}$ (4.82) with the transpose of the '11' block extended to the '22' block

$$\hat{\mathcal{H}}_{\hat{\mathfrak{W}}}(x_p) \;\approx\; \hat{S}\;\hat{\beta}^{aa}_{\hat{\mathfrak{W}}}\;\hat{\gamma}^{\mu,aa}_{\hat{\mathfrak{W}}}\;\boldsymbol{\hat{\partial}_{p,\mu}}-\imath\,\varepsilon_p\;\hat{1}_{2N_0\times 2N_0}\;; \tag{4.81}$$

$$\hat{G}^{(0)} \;=\; \big[\hat{\mathcal{H}}_{\hat{\mathfrak{W}}}(\hat{x}_p)\big]^{-1}=\begin{pmatrix}\hat{g}^{(0)} & \\ & \big[\hat{g}^{(0)}\big]^T\end{pmatrix}. \tag{4.82}$$

In compliance with the interaction picture, one has to calculate relations (4.83,4.84) for $\hat{g}^{(0)}_{N;M}$, $[\hat{g}^{(0)}]^T_{N;M}$ with colour-dressed gamma matrices but which fulfill the identical Clifford algebra as the original untransformed matrices $\hat{\beta}$, $\hat{\gamma}^{\mu}$

$$\hat{g}^{(0)}_{g,n,s;f,m,r} \;=\; \Big[\hat{\beta}^{11}_{\hat{\mathfrak{W}}}\;\hat{\gamma}^{\mu,11}_{\hat{\mathfrak{W}}}\;\boldsymbol{\hat{\partial}_{p,\mu}}-\imath\,\varepsilon_p\Big]^{-1}_{g,n,s;f,m,r}\;; \tag{4.83}$$

$$\big[\hat{g}^{(0)}\big]^T_{g,n,s;f,m,r} \;=\; \Big[-\hat{\beta}^{22}_{\hat{\mathfrak{W}}}\;\hat{\gamma}^{\mu,22}_{\hat{\mathfrak{W}}}\;\boldsymbol{\hat{\partial}_{p,\mu}}-\imath\,\varepsilon_p\Big]^{-1}_{g,n,s;f,m,r}\;. \tag{4.84}$$

We use the overview of the anomalous-doubled Hilbert space, summarized in appendix A, and the definitions and notations of chapter 4 in [11] so that the 'anti-unitary', 'anti-linear' '22' states accompany as extensions the original states in the '11' block. One therefore gains the Green functions (4.85,4.86) which can be combined into the anomalous-doubled one $\widehat{\langle x_p}^{a}|\hat{G}^{(0)}|\widehat{y_p\rangle}^{b}$ (4.87) with the anomalous-doubled states $\widehat{\langle x_p}^{a}|$, $|\widehat{y_p\rangle}^{b}$ (cf appendix A)

$$\langle x_p|\big(\hat{\beta}\hat{\gamma}^{\mu}\hat{\partial}_{p,\mu}-\imath\,\varepsilon_p\big)^{-1}|y_p\rangle \;=\; -\Big(\hat{\gamma}^{\mu}\frac{\partial}{\partial x^{\mu}_p}\hat{\beta}-\imath\,\varepsilon_p\Big)\frac{\theta_p(x^0_p-y^0_p)}{4\pi\,|\vec{x}-\vec{y}|}\;\delta\big(|x^0_p-y^0_p|-|\vec{x}-\vec{y}|\big)\;; \tag{4.85}$$

$$\begin{aligned}\overline{\langle x_p|}\big(\hat\beta\hat\gamma^\mu\hat\partial_{p,\mu}-\imath\,\varepsilon_p\big)^{T,-1}\overline{|y_p\rangle} &= \langle y_p|\big(\hat\beta\hat\gamma^\mu\hat\partial_{p,\mu}-\imath\,\varepsilon_p\big)^{-1}|x_p\rangle = \\ &= -\Big(\hat\gamma^\mu\frac{\partial}{\partial y_p^\mu}\hat\beta-\imath\,\varepsilon_p\Big)\frac{\theta_p(y_p^0-x_p^0)}{4\pi\,|\vec y-\vec x|}\;\delta\big(|y_p^0-x_p^0|-|\vec y-\vec x|\big)\;;\end{aligned} \tag{4.86}$$

$$\widehat{\langle x_p}^{a}|\hat G^{(0)}|\widehat{y_p\rangle}^{b} = -\delta_{ab}\Big(\hat S^a\;\hat\gamma^{\mu,aa}_{\hat{\mathfrak{W}}}\;\frac{\partial}{\partial x_p^\mu}\;\hat\beta^{aa}_{\hat{\mathfrak{W}}}-\imath\,\varepsilon_p\Big)\frac{\delta\big(|x_p^0-y_p^0|-|\vec x-\vec y|\big)}{4\pi\,|\vec x-\vec y|}\begin{pmatrix}\theta_p(x_p^0-y_p^0) & 0\\ 0 & \theta_p(y_p^0-x_p^0)\end{pmatrix}^{ab}. \tag{4.87}$$

Since we consider time contour Green functions (4.85-4.87), we include the generalized contour Heaviside function $\theta_p(x_p^0-y_p^0)$ (4.88) and also obtain the standard relation (4.89) of non-equilibrium Green functions [25]. Relation (4.89) incorporates the time ordering of fields and operators in the path integral (4.74-4.77) according to the time contour integrals with forward and backward propagation

$$\begin{aligned}\theta_{p=+}(x_+^0-y_+^0) &= \theta(x_+^0-y_+^0)\;; & x_+^0 &> y_+^0\\ \theta_{p=-}(x_-^0-y_-^0) &= \theta(y_-^0-x_-^0)\;; & y_-^0 &> x_-^0\end{aligned} \tag{4.88}$$

$$\widehat{\langle x_+}^{a}|\hat G^{(0)}|\widehat{y_+\rangle}^{a}+\widehat{\langle x_-}^{a}|\hat G^{(0)}|\widehat{y_-\rangle}^{a} = \widehat{\langle x_-}^{a}|\hat G^{(0)}|\widehat{y_+\rangle}^{a}\;;\quad \widehat{\langle x_+}^{a}|\hat G^{(0)}|\widehat{y_-\rangle}^{a}\equiv 0\;. \tag{4.89}$$

Corresponding to the interaction picture, the Green functions (4.85-4.87) are specialized onto the massless case of the standard Feynman propagator so that the chirality decouples into conserved helicity states as being 'exact' quantum numbers. (An observer of an arbitrary inertial system cannot 'overrun' the dressed massless BCS-states in the coset matrices $\hat T_{\hat{\mathfrak{W}}}(x_p)$, $\hat T^{-1}_{\hat{\mathfrak{W}}}(x_p)$ travelling on the light-cone '$\delta(|x_p^0-y_p^0|-|\vec x-\vec y|\,)$' so that the internal angular momentum cannot be projected onto the opposite momentum direction for a different helicity !). The anomalous-doubled Green function for the massles case (4.87) consists of two time contour Heaviside functions (4.88) $\theta_p(x_p^0-y_p^0)$, $\theta_p(y_p^0-x_p^0)$ on the light-cone '$\delta(|x_p^0-y_p^0|-|\vec x-\vec y|\,)$' which result into opposite time contour propagations $\eta_p(x_p^0-y_p^0)\;>0$, $\eta_p(y_p^0-x_p^0)\;>0$ (in the 'contour sense') concerning the '11' and '22' density blocks. Therefore, we separate the anomalous-doubled, time contour step functions from $\widehat{\langle x_p}^{a}|\hat G^{(0)}|\widehat{y_p\rangle}^{a}$ and introduce the Green function 'operator' $\widehat{\langle x_p}^{a}|\hat{\boldsymbol{\mathfrak{G}}}^{(\mathbf{0})}|\widehat{y_p\rangle}^{a}$ acting onto the anomalous-doubled, time contour Heaviside function $\Theta_p^{aa}(x_p^0-y_p^0)$

$$\begin{aligned}\widehat{\langle x_p}^{a}|\hat G^{(0)}|\widehat{y_p\rangle}^{b} &= -\delta_{ab}\Big(\hat S^a\;\hat\gamma^{\mu,aa}_{\hat{\mathfrak{W}}}\;\frac{\partial}{\partial x_p^\mu}\;\hat\beta^{aa}_{\hat{\mathfrak{W}}}-\imath\,\varepsilon_p\Big)\frac{\delta\big(|x_p^0-y_p^0|-|\vec x-\vec y|\big)}{4\pi\,|\vec x-\vec y|}\begin{pmatrix}\theta_p(x_p^0-y_p^0) & 0\\ 0 & \theta_p(y_p^0-x_p^0)\end{pmatrix}^{ab} \qquad (4.90)\\ &= \delta_{ab}\;\widehat{\langle x_p}^{a}|\hat{\boldsymbol{\mathfrak{G}}}^{(\mathbf{0})}|\widehat{y_p\rangle}^{a}\;\Theta_p^{aa}(x_p^0-y_p^0)\;;\\ \widehat{\langle x_p}^{a}|\hat{\boldsymbol{\mathfrak{G}}}^{(\mathbf{0})}|\widehat{y_p\rangle}^{a} &= -\delta_{ab}\Big(\hat S^a\;\hat\gamma^{\mu,aa}_{\hat{\mathfrak{W}}}\;\frac{\boldsymbol{\partial}}{\boldsymbol{\partial x_p^\mu}}\;\hat\beta^{aa}_{\hat{\mathfrak{W}}}-\imath\,\varepsilon_p\Big)\frac{\delta\big(|x_p^0-y_p^0|-|\vec x-\vec y|\big)}{4\pi\,|\vec x-\vec y|}\;; \qquad (4.91)\\ \Theta_p^{ab}(x_p^0-y_p^0) &= \delta_{ab}\begin{pmatrix}\theta_p(x_p^0-y_p^0) & 0\\ 0 & \theta_p(y_p^0-x_p^0)\end{pmatrix}^{ab}. \qquad (4.92)\end{aligned}$$

The anomalous-doubled Heaviside or time contour step function $\Theta_p^{ab}(x_p^0-y_p^0)$ restricts the possible terms in the gradient expansion of the action $\mathcal{A}_{DET}[\hat T_{\hat{\mathfrak{W}}};\hat\partial_{\hat{\mathfrak{W}}}]$ (4.75) because the trace operations in (4.75) also involve the contour extended traces of spacetime; in consequence one obtains as the remaining terms in the gradient expansion of $\mathcal{A}_{DET}[\hat T_{\hat{\mathfrak{W}}};\hat\partial_{\hat{\mathfrak{W}}}]$ (4.75) only those in which the anomalous-doubled, time contour step functions $\Theta_p^{aa}(x_p^0-y_p^0)$ do not result into contradictory propagations concerning the time contour extended ordering of gradient terms $\Delta\hat{\mathcal{H}}_{\hat{\mathfrak{W}}}(x_p)$ (4.80) (e.g. $\Theta_p^{aa}(x_p^0-y_p^0)\cdot\Theta_p^{aa}(y_p^0-x_p^0)\equiv 0$!). This restriction of terms is missing in the case of the gradient expansion of the action $\mathcal{A}_{J_{\psi;\hat{\mathfrak{W}}}}[\hat T_{\hat{\mathfrak{W}}};\hat\partial_{\hat{\mathfrak{W}}}]$ (4.76). In continuation of principles for a gradient expansion, we state that an anomalous-doubled field $\Psi_M^a(x_p)$ propagates with the block diagonal. doubled Green function $\widehat{\langle x_p}^{a}|\hat G^{(0)}|\widehat{y_q\rangle}^{b}$

$$\Psi_M^{a(=1,2)}(x_p) = \begin{pmatrix}\psi_M(x_p)\\ \psi_M^*(x_p)\end{pmatrix}^{a} = \int_C d^4y_q\;\mathcal{N}^2\,\widehat{\langle x_p}^{a}|\hat G^{(0)}_{M;N}|\widehat{y_q\rangle}^{b}\begin{pmatrix}\psi_N(y_q)\\ \psi_N^*(y_q)\end{pmatrix}^{b(=1,2)}. \tag{4.93}$$

This principle has to be used in the expansion of $\mathcal{A}_{J_{\psi;\hat{\mathfrak{W}}}}[\hat{T}_{\hat{\mathfrak{W}}};\hat{\partial}_{\hat{\mathfrak{W}}}]$ (4.76) where one starts to propagate with the source field $J^{a}_{\psi;M_1}(x_p)$ on the right-hand side of the action for a BEC wavefunction. It replaces the wavefunction $\Psi^{b}_{N}(y_q)$ in (4.93). However, the propagation of fields with (4.93) is not directly applicable for the action $\mathcal{A}_{DET}[\hat{T}_{\hat{\mathfrak{W}}};\hat{\partial}_{\hat{\mathfrak{W}}}]$ (4.75) because of the cyclic invariance of traces, both of the internal state spaces $\mathfrak{tr}_{N_f,\hat{\gamma}^{(\mu)}_{mn},N_c}$ and the Hilbert state trace of doubled quantum mechanics.

5 Derivation of a nontrivial topology and the chiral anomaly

5.1 Comparison to the Skyrme model with homotopy group $\Pi_3(\mathbf{SU}(2))=\mathsf{Z}$

The derived actions $\mathcal{A}_{DET}[\hat{T},\hat{\mathcal{V}};\hat{\partial}]$, $\mathcal{A}_{J_\psi}[\hat{T},\hat{\mathcal{V}};\hat{\partial}]$ (3.110-3.116) consist of coset matrices $\hat{T}(x_p)=\exp\{-\hat{Y}(x_p)\}$ whose manifold is determined by the coset space $\mathrm{SO}(N_0,N_0)\,/\,\mathrm{U}(N_0)$ with $N_0=(N_f=2)\times 4_{\hat{\gamma}}\times(N_c=3)=24$. Therefore, the question arises in comparison to effective nucleon models as the $\mathrm{SU}_L(2)\otimes\mathrm{SU}_R(2)\,/\,\mathrm{SU}_V(2)$ chiral Skyrme Lagrangian whether the mapping from 3(+1) spacetime to the $\mathrm{SO}(N_0,N_0)\,/\,\mathrm{U}(N_0)$ coset space also allows for a nontrivial homotopic classification of BCS terms within the $\mathrm{su}(N_0,N_0)\,/\,\mathrm{u}(N_0)$ coset generator [11].

Since the so-called 'σ-field' $\sigma(x_p)$ of the original 'σ-model' has turned out to be dependent on the other three pion fields $\vec{\pi}(x_p)=(\pi_1(x_p)\,,\pi_2(x_p)\,,\pi_3(x_p)\,)$ as a two pion resonance, one has to introduce the nonlinear condition

$$\big(\,\sigma(x_p)\,\big)^2+\vec{\pi}(x_p)\cdot\vec{\pi}(x_p)=1\;, \tag{5.1}$$

in order to remove the non-physical, dependent $\sigma(x_p)$ field degree of freedom from an effective Lagrangian. The nonlinear restriction (5.1) is analogous to the four dimensional $\vec{x}\cdot\vec{x}-(x^0)^2$, $\mathrm{SO}(3,1)$ Lorentz symmetry, but within the 'internal isospin space' which gives rise to the chiral symmetry $\mathrm{SU}_L(2)\otimes\mathrm{SU}_R(2)$ for an appropriate effective Lagrangian in the massless case. A suitable Lagrangian, which incorporates the nonlinear restriction for only three independent fields, is given by the Skyrme Lagrangian which only comprises quadratic and quartic derivatives of axial $\mathrm{SU}_A(2)$ isospin matrices $\hat{U}(x_p)$ corresponding to stable, static energy configurations in 3+1 spacetime dimensions (compare [17, 18, 19] concerning the original Skyrme model with Lagrangian (5.4) and 'Derrick's theorem' [13] for stable, static energy configurations in the 3(+1) spacetime)

$$\begin{aligned}
\hat{U}(x_p) &= \exp\big\{\imath\,\vec{\tau}\cdot\vec{\varphi}(x_p)\big\}\;;\qquad \hat{U}^{\dagger}(x_p)\,\hat{U}(x_p)=\hat{1}\;; && (5.2)\\
\hat{L}_\mu(x_p) &= \hat{U}^{\dagger}(x_p)\,\big(\hat{\partial}_{p,\mu}\hat{U}(x_p)\big)\;; && (5.3)\\
\mathcal{L}_{Skyrme} &= -\frac{f_\pi^2}{4}\,\underset{N_f}{\mathfrak{tr}}\Big(\hat{L}_\mu(x_p)\,\hat{L}^{\mu}(x_p)\Big)+\frac{1}{32\,e^2}\,\underset{N_f}{\mathfrak{tr}}\Big(\big[\hat{L}^{\mu}(x_p)\,,\hat{L}^{\nu}(x_p)\big]_{-}\big[\hat{L}_\mu(x_p)\,,\hat{L}_\nu(x_p)\big]_{-}\Big)\;; && (5.4)\\
f_\pi &\mathrel{\hat{=}} \text{'pion decay constant'}\;;\quad e\mathrel{\hat{=}}\text{'dimensionless constant' for size of Skyrmion}\;; && (5.5)\\
\vec{\tau} &\mathrel{\hat{=}} \text{'isospin Pauli matrices'}\;.
\end{aligned}$$

The axial $\mathrm{SU}_A(2)$ isospin matrices $\hat{U}(x_p)=\exp\{\imath\,\vec{\tau}\cdot\vec{\varphi}(x_p)\}$ include the nonlinear restriction (5.1) due to the reduction to three independent, internal angle field degrees of freedom $\vec{\varphi}(x_p)=(\,\varphi_1(x_p)\,,\varphi_2(x_p)\,,\varphi_3(x_p)\,)$. The $\mathrm{SU}_A(2)$ isospin matrices $\hat{U}(x_p)$ determine the four dependent isospin fields $(\,\sigma(x_p)\,,\vec{\pi}(x_p)\,)$

$$\begin{aligned}
\hat{U}(x_p) &= \exp\{\imath\,\vec{\tau}\cdot\vec{\varphi}(x_p)\}=\sigma(x_p)+\imath\,\vec{\pi}(x_p)\;; && (5.6)\\
\sigma(x_p) &= \hat{1}_{2\times 2}\cdot\cos\big(|\vec{\varphi}(x_p)|\big)\;;\qquad \vec{\pi}(x_p)=\vec{\tau}\cdot\frac{\vec{\varphi}(x_p)}{|\vec{\varphi}(x_p)|}\cdot\sin\big(|\vec{\varphi}(x_p)|\big)\;;\\
|\vec{\varphi}(x_p)| &= \sqrt{\varphi_1^2(x_p)+\varphi_2^2(x_p)+\varphi_3^2(x_p)}\;,
\end{aligned}$$

and give rise to the homotopic classification $\Pi_3(\mathrm{SU}(2)\,)=\mathsf{Z}$ for topological solitons following from the mapping of the compactified three dimensional coordinate space to the internal $\mathrm{SU}(2)\sim\mathrm{S}^3$ isospin space or sphere. This homotopic

[11] The isospin chiral symmetry $\mathrm{SU}_L(2)\otimes\mathrm{SU}_R(2)$ is spontaneously broken by the vector isospin invariance $\mathrm{SU}_V(2)$ of the vacuum states with the appearance of three massless pseudoscalar Nambu-Goldstone bosons or the pions $\pi^0(x_p)$, $\pi^{\pm}(x_p)$ [26, 27].

classification $\Pi_3(\mathrm{SU}(2)\,) = \mathsf{Z}$ with nontrivial winding numbers is completely independent from the Skyrme Lagrangian (5.2-5.5) with the quadratic and quartic derivatives of the $\mathrm{SU}_A(2)$ isospin matrices $\hat{U}(x_p)$. The zero component B^0 of the integrated, topological current density $B^\mu(x_p)$, corresponding to $\Pi_3(\mathrm{SU}(2)\,) = \mathsf{Z}$, is assigned to the Baryon number for the prevailing $\mathrm{SU}_A(2)$ isospin field configuration $\hat{U}(x_p)$

$$\begin{aligned} B^\mu(x_p) &= \frac{\varepsilon^{\mu\nu\kappa\lambda}}{24\,\pi^2} \underset{N_f}{\mathfrak{tr}} \Big[\hat{L}_\nu(x_p)\,\hat{L}_\kappa(x_p)\,\hat{L}_\lambda(x_p)\Big]\;; & (5.7)\\ \big(\hat{\partial}_{p,\mu} B^\mu(x_p)\big) &= 0\;; & (5.8)\\ B^0 &= \int_{\mathsf{R}^3} d^3x\;\; B^0(x_p) \;\rightarrow\; \text{Baryon number}\;. & (5.9)\end{aligned}$$

However, this interpretation and the conserved topological current $(\hat{\partial}_{p,\mu} B^\mu(x_p)\,) = 0$ as a Baryon current is completely independent from the Skyrme Lagrangian and is argued to occur as an extension of QCD in the large N_c limit ($\mathrm{SU}(N_c \to \infty)$) of colour degrees of freedom [28, 29, 30]. This causes the question whether the presented approach of previous sections with fixed $\mathrm{SU}_c(N_c = 3)$ colour symmetry also contains nontrivial topologies within the derived effective actions $\mathcal{A}_{DET}[\hat{T}, \langle\hat{\mathcal{V}}\rangle_{(3.59)}; \hat{\mathcal{J}}]$, $\mathcal{A}_{J_\psi}[\hat{T}, \langle\hat{\mathcal{V}}\rangle_{(3.59)}; \hat{\mathcal{J}}]$ of coset matrices and generators $\hat{T}(x_p) = \exp\{-\hat{Y}(x_p)\,\}$ with anti-symmetric, even-valued BCS quark pairs $\hat{X}_{N;M}(x_p)$, $\hat{X}^\dagger_{N;M}(x_p)$ (3.72-3.78).

5.2 Instantons of BCS terms from the Hopf mapping $\Pi_3(S^2) = \mathsf{Z}$ and the Hopf invariant

The anti-symmetric, even-valued BCS terms $\hat{X}_{N;M}(x_p)$, $\hat{X}^\dagger_{N;M}(x_p)$ are given by a non-compact, internal manifold with 'hyperbolic' trigonometric functions and allow to achieve a classification of 'instantons' from the derived actions $\mathcal{A}_{DET}[\hat{T}, \langle\hat{\mathcal{V}}\rangle_{(3.59)}; \hat{\mathcal{J}}]$, $\mathcal{A}_{J_\psi}[\hat{T}, \langle\hat{\mathcal{V}}\rangle_{(3.59)}; \hat{\mathcal{J}}]$ instead of the topological solitons as in the Skyrme model. Furthermore, we recognize that the eigenvalues, as the crucial elements of $\hat{X}_{N;M}(x_p)$, $\hat{X}^\dagger_{N;M}(x_p)$, are determined by the quaternion-valued, complex field variables $\overline{f}_{\overline{M}}(x_p)$ (3.74-3.78) with anti-symmetric Pauli matrix $(\hat{\tau}_2)_{gf}$ of isospin space. Therefore, one can only accomplish a mapping from the compactified three dimensional coordinate space $\vec{x}$ to two independent, real field degrees of freedom within the internal isospin space; in consequence, a valid classification according to the Hopf fibration $\Pi_3(\mathrm{S}^2) = \mathsf{Z}$ may be expected for the mapping from the compactified three dimensional coordinate space to the complex, quaternionic eigenvalues with anti-symmetric Pauli-matrix $(\hat{\tau})_{gf}$. We also anticipate a Hopf classification following from $\Pi_{\mathrm{S}^{2n-1}}(\mathrm{S}^n)$ with $n = 4$ (instead of $n = 2$ as for the eigenvalues) because the diagonalizing, eigenvector matrices $\hat{P}^{aa}_{N_0\times N_0}(x_p)$ (3.79-3.83) of $\hat{X}_{N;M}(x_p)$, $\hat{X}^\dagger_{N;M}(x_p)$ are specified by four complex-valued fields for the four independent quaternions with 2×2 Pauli-matrices $(\hat{\tau}_0\,,\,\vec{\tau})$ as the basic entries along the off-diagonal matrix elements of $\hat{\mathcal{G}}_{D;f\overline{M};g\overline{M}}(x_p)$ (3.79-3.83) [15]. However, it is questionable whether an additional Hopf fibration of $\Pi_{\mathrm{S}^{15}}(\mathrm{S}^8)$ (or $n = 8$) can be realized in our model of coset matrices $\hat{T}(x_p)$ with BCS terms because this involves 'octonions' instead of quaternions so that associativity is not preserved.

We verify the assumption of a valid Hopf mapping $\Pi_3(\mathrm{S}^2) = \mathsf{Z}$ from the three dimensional coordinate space to an anti-symmetric, complex eigenvalue matrix $(\hat{\tau}_2)_{gf}$ with fields $\overline{f}_r(x_p)$ $(r = 1, \ldots, N_c = 3)$, $(\overline{f}_r(x_p) \in \mathsf{C}_{\mathsf{even}})$

$$\hat{X}_{N;M}(x_p) = \hat{X}_{g,\overline{N};f,\overline{M}}(x_p) = (\hat{\tau}_2)_{gf}\;\overline{f}_r(x_p)\;\delta_{\overline{N};\overline{M}} = (\hat{\tau}_2)_{gf}\;\overline{f}_r(x_p)\;\delta_{sr}\;\delta_{nm}\;, \tag{5.10}$$

by using the axial current relation of quark fields with the chiral anomaly (cf. the derivation in appendix C)

$$\begin{aligned} &\boldsymbol{\hat{\partial}_{p,\mu}\Big(\bar{\psi}(x_p)\,\hat{\gamma}^\mu\,\hat{\gamma}_5\,\hat{\mathfrak{t}}_0\,\psi(x_p)\Big) = -\frac{\varepsilon^{\kappa\lambda\mu\nu}}{16\,\pi^2}\underset{N_f,N_c}{\mathfrak{tr}}\Big[\hat{\mathfrak{t}}_0\,\hat{F}_{\kappa\lambda}(x_p)\,\hat{F}_{\mu\nu}(x_p)\Big] + \bar{\psi}(x_p)\,\hat{\gamma}_5\,\big\{\hat{\mathfrak{t}}_0\,,\,\hat{m}\big\}_\dagger\,\psi(x_p) +}\\ &+\;\; j^\dagger_\psi(x_p)\,\hat{\gamma}_5\,\hat{\mathfrak{t}}_0\,\psi(x_p) - \psi^\dagger(x_p)\,\hat{\gamma}_5\,\hat{\mathfrak{t}}_0\,j_\psi(x_p) + \frac{1}{2}\Big[\psi^T(x_p)\Big(\hat{j}^\dagger_{\psi\psi}(x_p)\,\hat{\gamma}_5\,\hat{\mathfrak{t}}_0 + \hat{\gamma}_5\,\hat{\mathfrak{t}}_0^T\,\hat{j}^\dagger_{\psi\psi}(x_p)\Big)\psi(x_p) +\\ &-\;\; \psi^\dagger(x_p)\Big(\hat{j}_{\psi\psi}(x_p)\,\hat{\gamma}_5\,\hat{\mathfrak{t}}_0^* + \hat{\gamma}_5\,\hat{\mathfrak{t}}_0\,\hat{j}_{\psi\psi}(x_p)\Big)\psi^*(x_p)\Big] + \frac{1}{2}\int_C d^4y_q\;\times \qquad\qquad (5.11)\\ &\times\;\; \Bigg[\Psi^\dagger(y_q,x_p)\,\hat{\mathcal{J}}(y_q,x_p)\begin{pmatrix}\hat{\gamma}_5\,\hat{\mathfrak{t}}_0 & \\ & -\hat{\gamma}_5\,\hat{\mathfrak{t}}_0^*\end{pmatrix}\Psi(x_p) + \Psi^\dagger(x_p)\begin{pmatrix}-\hat{\gamma}_5\,\hat{\mathfrak{t}}_0 & \\ & \hat{\gamma}_5\,\hat{\mathfrak{t}}_0^T\end{pmatrix}\hat{\mathcal{J}}(x_p,y_q)\,\Psi(y_q)\Bigg]\,. \end{aligned}$$

We have included the various source fields of $\mathcal{A}_S[\hat{\jmath}, J_\psi, \hat{J}_{\psi\psi}, \hat{\jmath}^{(\hat{F})}]$ (2.22) and a finite mass term $\hat{m}$ within the axial symmetry variations of the original QCD-type path integral (2.25-2.27) with the constant, hermitian isospin matrix $(\hat{\mathfrak{t}}_0)_{gf}$. As one performs the zero-mass limit $\hat{m} \to 0$ for vanishing source fields, the two chiral states decouple and result into conserved helicity states of massless fermions moving on the light-cone so that their projection of spin onto the momentum cannot be overrun by other observers for a different helicity and becomes conserved. However, this axial current relation also contains the chiral anomaly (Adler-Bell-Jackiw anomaly, first term on the right-hand side of (5.11)) apart from the symmetry violating mass and source field terms. This anomaly is obtained from the calculation of the Jacobian for quark fields under an axial chiral transformation with a gauge invariant cut-off regulator [31, 32, 33]. This chiral anomaly has nontrivial instanton numbers and can also be rewritten in terms of a conserved Chern-Simons current $K^\mu(x_p)$

$$\begin{aligned}
(N_I \in \mathsf{Z}\,,\text{ instanton number}) &= \frac{\varepsilon^{\mu\nu\kappa\lambda}}{32\,\pi^2}\int_{-\infty}^{+\infty} d^4x_p \;\underset{N_c}{\mathfrak{tr}}\Big[\hat{F}_{\mu\nu}(x_p)\;\hat{F}_{\kappa\lambda}(x_p)\Big]\;; && (5.12)\\
\big(\hat{\partial}_{p,\mu}K^\mu(x_p)\big) &= \frac{\varepsilon^{\mu\nu\kappa\lambda}}{8\,\pi^2}\Big(\hat{\partial}_{p,\mu}\underset{N_c}{\mathfrak{tr}}\Big[\hat{A}_\nu(x_p)\;\big(\hat{\partial}_{p,\kappa}\hat{A}_\lambda(x_p)\big) - \frac{2\,\imath}{3}\;\hat{A}_\nu(x_p)\;\hat{A}_\kappa(x_p)\;\hat{A}_\lambda(x_p)\Big]\Big) && (5.13)\\
&= \frac{\varepsilon^{\mu\nu\kappa\lambda}}{32\,\pi^2}\;\underset{N_c}{\mathfrak{tr}}\Big[\hat{F}_{\mu\nu}(x_p)\;\hat{F}_{\kappa\lambda}(x_p)\Big]\;.
\end{aligned}$$

If we multiply (5.11) by the factor one-half for a spin '$\frac{1}{2}\,\hbar$' angular momentum and perform the four dimensional spacetime integration in the massless limit with vanishing source fields, one attains on the right-hand side the integer instanton numbers and on the left-hand side the integrated, zero-helicity component for vanishing current densities at spatial infinity from the Gaussian integration law

$$\int d^3\vec{x}\;\;\psi^\dagger(x_p)\;\hat{\beta}\;\frac{1}{2}\;\hat{\gamma}^0\;\hat{\gamma}_5\;\hat{\mathfrak{t}}_0\;\psi(x_p)\;\Big|_{x_p^0=-\infty}^{x_p^0=+\infty} \;=\; (N_I \in \mathsf{Z}\,,\text{ instanton number})\;\cdot\;(N_f = 2)\;. \qquad (5.14)$$

The last, integrated relation (5.14) of a conserved helicity number is therefore quantized according to the instanton numbers from the chiral anomaly. (Eq. (5.14) contains two isospin degrees of freedom ($N_f = 2$) with isospin unit matrix $(\hat{\mathfrak{t}}_0)_{gf} = (\hat{1}_{2\times2})_{gf}$). Since we derive the Hopf invariant $(V_{S^{2n-1}})^{-1}\int_{\mathrm{S}^{2n-1}}\hat{\omega}_{n-1}\wedge(d\hat{\omega}_{n-1})$ $(n=2)$ from relation (5.14) by suitable differentiation of the transformed path integrals with respect to the source field $\hat{\jmath}^{ba}_{N:M}(y_q, x_p)$, the derived Hopf invariant (for the massless case) classifies the prevailing field configuration of the coset matrix $\hat{T}(x_p)$ with BCS terms according to their content as a conserved, integrated helicity number.

In consequence, one has to transform relation (5.14) to terms of coset matrices $\hat{T}(x_p)$ and has to find a Hopf invariant $(V_{S^{2n-1}})^{-1}\int_{\mathrm{S}^{2n-1}}\hat{\omega}_{n-1}\wedge(d\hat{\omega}_{n-1})$ $(n=2)$ of a one-form $\hat{\omega}_1(x_p)$ within the actions $\mathcal{A}_{DET}[\hat{T}, \langle\hat{\mathcal{V}}\rangle_{(3.59)}; \hat{\jmath}]$, $\mathcal{A}_{J_\psi}[\hat{T}, \langle\hat{\mathcal{V}}\rangle_{(3.59)}; \hat{\jmath}]$ for the specified eigenvalue sort $(\hat{\tau}_2)_{gf}\;\overline{f}_r(x_p)\;\delta_{\overline{N:M}}$ of the coset generator $\hat{X}_{N;M}(x_p)$ $(r = 1, \ldots, N_c = 3)$. We apply the particular representation of Dirac gamma matrices of [20]

$$\hat{\beta} = \begin{pmatrix} 0 & \hat{1}_{2\times2} \\ \hat{1}_{2\times2} & 0 \end{pmatrix}\;;\quad \hat{\gamma}^0 = -\imath\,\hat{\beta}\;;\quad \vec{\gamma} = -\imath\begin{pmatrix} 0 & \vec{\sigma} \\ -\vec{\sigma} & 0 \end{pmatrix}\;;\quad \hat{\gamma}_5 = -\imath\,\hat{\gamma}^0\,\hat{\gamma}^1\,\hat{\gamma}^2\,\hat{\gamma}^3 = \begin{pmatrix} \hat{1}_{2\times2} & 0 \\ 0 & -\hat{1}_{2\times2} \end{pmatrix}\;, \qquad (5.15)$$

and perform an anomalous doubling of quark fields $\psi_M(x_p) \to \Psi^a_M(x_p) = \{\psi_M(x_p)\,,\,\psi^*_M(x_p)\}$ with anomalous doubled 'Gamma' matrices $\hat{\mathfrak{B}}$, $\hat{\Gamma}^\mu$, $\hat{\Gamma}_5$ for the left-hand side of the original, axial current relation (5.11) with the chiral anomaly

$$\begin{aligned}
\Big(\hat{\partial}_{p,\mu}\psi^\dagger(x_p)\;\hat{\beta}\;\hat{\gamma}^\mu\;\hat{\gamma}_5\;\hat{\mathfrak{t}}_0\;\psi(x_p)\Big) &= \frac{1}{2}\;\hat{\partial}_{p,\mu}\left[\Psi^{\dagger,b}(x_p)\;\hat{S}\begin{pmatrix} \hat{\beta}\,\hat{\gamma}^\mu\,\hat{\gamma}_5 & 0 \\ 0 & \big(\hat{\beta}\,\hat{\gamma}^\mu\,\hat{\gamma}_5\big)^T \end{pmatrix}^{ba}\Psi^a(x_p)\right] = && (5.16)\\
= -\frac{1}{2}\;\hat{\partial}_{p,\mu}&\Bigg(\underset{N_f,\hat{\gamma}^{(\mu)}_{mn},N_c}{\overset{a(=1,2)}{\mathfrak{Tr}}}\Big[\Big(\hat{S}\;\hat{\mathfrak{B}}\;\hat{\Gamma}^\mu\;\hat{\Gamma}_5\Big)^{ba}\;\Psi^a(x_p)\otimes\Psi^{\dagger,b}(x_p)\Big]\Bigg)\;;\quad \big((\hat{\mathfrak{t}}_0)_{gf} = (\hat{1}_{2\times2})_{gf}\big)\;;
\end{aligned}$$

$$\hat{\mathfrak{B}} = \begin{pmatrix} \hat{\beta} & 0 \\ 0 & \hat{\beta}^T \end{pmatrix}\;;\quad \hat{\mathfrak{B}}\,\hat{\Gamma}^\mu = \begin{pmatrix} \hat{\beta}\,\hat{\gamma}^\mu & 0 \\ 0 & \big(\hat{\beta}\,\hat{\gamma}^\mu\big)^T \end{pmatrix}\;;\quad \hat{\Gamma}_5 = \begin{pmatrix} \hat{\gamma}_5 & 0 \\ 0 & (\hat{\gamma}_5^T = \hat{\gamma}_5) \end{pmatrix}\;. \qquad (5.17)$$

The scalar product of anomalous doubled quark fields is converted to a dyadic product in an anomalous doubled, 'internal space' trace relation so that we can relate the dyadic product of doubled quark fields to the anomalous doubled self-energy matrix. Furthermore, we can track the scalar or dyadic product of quark fields in (5.16) by differentiation of the generating functions with respect to the source $\hat{\mathfrak{J}}_{N;M}^{ba}(x_p, x_p)$ to corresponding terms of coset matrices; thus, the axial current relation of the original quark fields is transformed to a relation determined by coset matrices within the path integral (3.110-3.116)

$$\hat{\partial}_{p,\mu}\Big(\psi^\dagger(x_p)\;\hat{\beta}\;\hat{\gamma}^\mu\;\hat{\gamma}_5\;\hat{\mathfrak{t}}_0\;\psi(x_p)\Big) = -\frac{1}{2}\;\hat{\partial}_{p,\mu}\Bigg(\underset{N_f,\hat{\gamma}_{mn}^{(\mu)},N_c}{\overset{a(=1,2)}{\mathfrak{Tr}}}\Big[\Big(\hat{S}\;\hat{\mathcal{B}}\;\hat{\Gamma}^\mu\;\hat{\Gamma}_5\Big)_{N;M}^{ba}\;\Psi_M^a(x_p)\otimes\Psi_N^{\dagger,b}(x_p)\Big]\Bigg) \tag{5.18}$$
$$= \imath\,\mathcal{N}^2\;\hat{\partial}_{p,\mu}\Bigg(\underset{N_f,\hat{\gamma}_{mn}^{(\mu)},N_c}{\overset{a(=1,2)}{\mathfrak{Tr}}}\Big[\Big(\hat{S}\;\hat{\mathcal{B}}\;\hat{\Gamma}^\mu\;\hat{\Gamma}_5\Big)_{N;M}^{ba}\Big(\frac{\partial}{\partial\hat{\mathfrak{J}}_{N;M}^{ba}(x_p,x_p)}Z[\hat{\mathfrak{J}},J_\psi,\hat{J}_{\psi\psi},\hat{\mathfrak{j}}^{(\hat{F})}]\Big)\Big]\Bigg)\,.$$

We restrict to the 'interaction representation' of the action $\mathcal{A}_{DET}[\hat{T}_{\hat{\mathfrak{W}}};\hat{\mathfrak{J}}]$ of the determinant and neglect terms from $\mathcal{A}_{J_\psi}[\hat{T}_{\hat{\mathfrak{W}}};\hat{\mathfrak{J}}]$ having arbitrary-valued, anti-commuting source fields $J_\psi(x_p)$. This allows to extract a third order derivative term of $\Delta\hat{\mathcal{H}}_{\hat{\mathfrak{W}}}(\hat{T}_{\hat{\mathfrak{W}}}^{-1},\hat{T}_{\hat{\mathfrak{W}}}) = \hat{T}_{\hat{\mathfrak{W}}}^{-1}\;\hat{\mathcal{H}}_{\hat{\mathfrak{W}}}\;\hat{T}_{\hat{\mathfrak{W}}} - \hat{\mathcal{H}}_{\hat{\mathfrak{W}}}$ from $\mathcal{A}_{DET}[\hat{T}_{\hat{\mathfrak{W}}};\hat{\mathfrak{J}}]$ by differentiation with respect to $\hat{\mathfrak{J}}_{N;M}^{ba}(x_p,x_p)$

$$Z[\hat{\mathfrak{J}}_{\hat{\mathfrak{W}}},J_{\psi;\hat{\mathfrak{W}}},\hat{J}_{\psi\psi}] = \int d[\hat{T}^{-1}(x_p)\;d\hat{T}(x_p)]\;\;\big\langle Z_{\hat{J}_{\psi\psi}}[\hat{\mathfrak{W}}\;\hat{T}_{\hat{\mathfrak{W}}}\;\hat{\mathfrak{W}}^{-1}]\big\rangle\;\times \tag{5.19}$$
$$\times\;\;\exp\Big\{\mathcal{A}_{DET}[\hat{T}_{\hat{\mathfrak{W}}};\hat{\mathfrak{J}}_{\hat{\mathfrak{W}}}\equiv 0] + \imath\,\mathcal{A}_{J_{\psi;\hat{\mathfrak{W}}}}[\hat{T}_{\hat{\mathfrak{W}}};\hat{\mathfrak{J}}_{\hat{\mathfrak{W}}}\equiv 0]\Big\}\;\times$$
$$\times\;\;\exp\Big\{-\frac{1}{2}\;\underset{\int_C d^4x_p\;\eta_p}{\overset{a(=1,2)}{\mathrm{TR}}}\;\underset{N_f,\hat{\gamma}_{mn}^{(\mu)},N_c}{\mathfrak{tr}}\Big[\widetilde{\mathfrak{J}}_{\hat{\mathfrak{W}}}(\hat{T}_{\hat{\mathfrak{W}}}^{-1},\hat{T}_{\hat{\mathfrak{W}}})\;\hat{\mathcal{H}}_{\hat{\mathfrak{W}}}^{\mathbf{-1}}\;\Big(\Delta\hat{\mathcal{H}}_{\hat{\mathfrak{W}}}(\hat{T}_{\hat{\mathfrak{W}}}^{-1},\hat{T}_{\hat{\mathfrak{W}}})\;\;\hat{\mathcal{H}}_{\hat{\mathfrak{W}}}^{\mathbf{-1}}\Big)^3\Big]\Big\} =$$
$$= \int d[\hat{T}^{-1}(x_p)\;d\hat{T}(x_p)]\;\;\big\langle Z_{\hat{J}_{\psi\psi}}[\hat{\mathfrak{W}}\;\hat{T}_{\hat{\mathfrak{W}}}\;\hat{\mathfrak{W}}^{-1}]\big\rangle\;\times$$
$$\times\;\;\exp\Big\{\mathcal{A}_{DET}[\hat{T}_{\hat{\mathfrak{W}}};\hat{\mathfrak{J}}_{\hat{\mathfrak{W}}}\equiv 0] + \imath\,\mathcal{A}_{J_{\psi;\hat{\mathfrak{W}}}}[\hat{T}_{\hat{\mathfrak{W}}};\hat{\mathfrak{J}}_{\hat{\mathfrak{W}}}\equiv 0]\Big\}\;\times$$
$$\times\;\;\Bigg(\Big(-\frac{1}{2}\Big)\;\underset{\int_C d^4x_p\;\eta_p}{\overset{a(=1,2)}{\mathrm{TR}}}\;\underset{N_f,\hat{\gamma}_{mn}^{(\mu)},N_c}{\mathfrak{tr}}\Big[\hat{T}_{\hat{\mathfrak{W}}}^{-1}\;\hat{I}\;\hat{S}\;\hat{\eta}\;\hat{\mathfrak{W}}^{-1}\;\frac{\hat{\mathfrak{J}}}{\mathcal{N}}\;\hat{\mathfrak{W}}\;\hat{\eta}\;\hat{S}\;\hat{I}\;\hat{T}_{\hat{\mathfrak{W}}}\;\hat{\mathcal{H}}_{\hat{\mathfrak{W}}}^{\mathbf{-1}}\Big(\Delta\hat{\mathcal{H}}_{\hat{\mathfrak{W}}}(\hat{T}_{\hat{\mathfrak{W}}}^{-1},\hat{T}_{\hat{\mathfrak{W}}})\;\;\hat{\mathcal{H}}_{\hat{\mathfrak{W}}}^{\mathbf{-1}}\Big)^3\Big]\Bigg)\,.$$

This yields following equation of the axial current in terms of BCS quark pairs instead of the original anti-commuting fields

$$\hat{\partial}_{p,\mu}\Big(\psi^\dagger(x_p)\;\hat{\beta}\;\hat{\gamma}^\mu\;\hat{\gamma}_5\;\hat{\mathfrak{t}}_0\;\psi(x_p)\Big) = \imath\;\Big(-\frac{1}{2}\Big)\;\hat{\partial}_{p,\mu}\Bigg(\underset{N_f,\hat{\gamma}_{mn}^{(\mu)},N_c}{\overset{a(=1,2)}{\mathfrak{Tr}}}\Big[\Big(\hat{S}\;\hat{\mathcal{B}}\;\hat{\Gamma}^\mu\;\hat{\Gamma}_5\Big)_{N;M}^{ba}\times \tag{5.20}$$
$$\times\;\;\Big(\hat{\mathfrak{W}}(x_p)\;\hat{S}\;\hat{I}\;\hat{T}_{\hat{\mathfrak{W}}}(x_p)\;\Big\langle\widehat{x_p}\Big|\hat{\mathcal{H}}_{\hat{\mathfrak{W}}}^{\mathbf{-1}}\Big(\Delta\hat{\mathcal{H}}_{\hat{\mathfrak{W}}}(\hat{T}_{\hat{\mathfrak{W}}}^{-1},\hat{T}_{\hat{\mathfrak{W}}})\;\;\hat{\mathcal{H}}_{\hat{\mathfrak{W}}}^{\mathbf{-1}}\Big)^3\Big|\widehat{x_p}\Big\rangle\;\;\hat{T}_{\hat{\mathfrak{W}}}^{-1}(x_p)\;\hat{I}\;\hat{S}\;\hat{\mathfrak{W}}^{-1}(x_p)\Big)_{M;N}^{ab}\Big]\Bigg)\,.$$

The 'interaction transformed' mean field operator $\hat{\mathcal{H}}_{\hat{\mathfrak{W}}}^{\mathbf{-1}}$ (4.55-4.57) of the propagation and the 'relative' gradient operator $\Delta\hat{\mathcal{H}}_{\hat{\mathfrak{W}}}(\hat{T}_{\hat{\mathfrak{W}}}^{-1},\hat{T}_{\hat{\mathfrak{W}}})$ are considered for the bulk of a nucleus so that we can approximate all 'interaction transformed' Dirac gamma matrices $\hat{\mathcal{B}}_{\hat{\mathfrak{W}}}$, $\hat{\Gamma}_{\hat{\mathfrak{W}}}^\mu$, $\hat{\Gamma}_{5,\hat{\mathfrak{W}}}$ with their original matrices apart from a constant similarity transformation. According to our ansatz (5.21-5.23) for $\hat{T}(x_p)$, the coset matrix is also not effected by the transformation to the 'interaction picture' because the transforming block diagonal matrices $\hat{\mathfrak{W}}_{N;M}^{aa}(x_p)$ need not be considered for constant similarity transformations in the bulk of a nucleus and decouple from the isospin space with indices 'g, f' for vanishing mass term $\hat{m}\to 0$

(compare (4.55-4.57))

$$\begin{aligned}
\Delta\hat{\mathcal{H}}_{\hat{\mathfrak{W}}}(\hat{T}_{\hat{\mathfrak{W}}}^{-1},\hat{T}_{\hat{\mathfrak{W}}}) &= \hat{T}_{\hat{\mathfrak{W}}}^{-1}\,\hat{\mathcal{B}}_{\hat{\mathfrak{W}}}(x_p)\,\hat{S}\,\hat{\Gamma}_{\hat{\mathfrak{W}}}^{\mu}(x_p)\,\hat{\boldsymbol{\partial}}_{\boldsymbol{p,\mu}}\,\hat{T}_{\hat{\mathfrak{W}}} - \hat{\mathcal{B}}_{\hat{\mathfrak{W}}}(x_p)\,\hat{S}\,\hat{\Gamma}_{\hat{\mathfrak{W}}}^{\mu}(x_p)\,\hat{\boldsymbol{\partial}}_{\boldsymbol{p,\mu}} && (5.21)\\
&\simeq \hat{T}^{-1}\,\hat{\mathcal{B}}\,\hat{S}\,\hat{\Gamma}^{\mu}\,\hat{\boldsymbol{\partial}}_{\boldsymbol{p,\mu}}\,\hat{T} - \hat{\mathcal{B}}\,\hat{S}\,\hat{\Gamma}^{\mu}\,\hat{\boldsymbol{\partial}}_{\boldsymbol{p,\mu}}\;;\\
\hat{T}_{\hat{\mathfrak{W}}}(x_p) &= \exp\big\{-\hat{Y}_{\hat{\mathfrak{W}}}(x_p)\big\}\;;\quad \hat{Y}_{\hat{\mathfrak{W}}}(x_p)=\begin{pmatrix}0 & \hat{X}_{\hat{\mathfrak{W}}}(x_p)\\ \hat{X}_{\hat{\mathfrak{W}}}^{\dagger}(x_p) & 0\end{pmatrix}\;; && (5.22)\\
\hat{X}_{\hat{\mathfrak{W}};N;M}(x_p) &= \hat{X}_{\hat{\mathfrak{W}};g,\overline{N};f,\overline{M}}(x_p)\simeq\big(\hat{\tau}_2\big)_{gf}\,\overline{f}_r(x_p)\,\delta_{\overline{N};\overline{M}}=\hat{X}_0(x_p)\;;\quad (r=1,\ldots,N_c=3)\;; && (5.23)\\
\Longrightarrow \hat{T}_{\hat{\mathfrak{W}}}(x_p) &\simeq \hat{T}_0(x_p)=\exp\left\{-\begin{pmatrix}0 & \hat{X}_0(x_p)\\ \hat{X}_0^{\dagger}(x_p) & 0\end{pmatrix}\right\}\;. && (5.24)
\end{aligned}$$

We introduce the anomalous doubled, diagonal spin matrices $\hat{\mathbb{S}}^k$ $(k=1,2,3)$ instead of the ordinary Pauli spin matrices $\hat{\sigma}^k$ through inclusion of the Dirac gamma matrix $\hat{\Gamma}_5$

$$\imath\,\hat{\mathbb{S}}^k=\hat{\mathcal{B}}\,\hat{\Gamma}^k\,\hat{\Gamma}_5=\imath\begin{pmatrix}\big(\hat{\mathbb{S}}^k\big)^{11} & \\ & \big(\hat{\mathbb{S}}^k\big)^{22}\end{pmatrix}=\imath\begin{pmatrix}\begin{pmatrix}\sigma^k & \\ & \sigma^k\end{pmatrix} & \\ & \begin{pmatrix}\sigma^k & \\ & \sigma^k\end{pmatrix}^T\end{pmatrix}, \tag{5.25}$$

and substitute this into the spacetime integrated, zero component of the axial current relation (5.20) for vanishing spatial current densities at the surface of the considered three volume. Since the block diagonal, anomalous doubled spin matrices $\hat{\mathbb{S}}^k$ (5.25) commute with the coset matrix $\hat{T}_0(x_p)$ of the internal isospin and colour degrees of freedom, the total trace $\mathfrak{Tr}_{N_f,\hat{\gamma}^{(\mu)},N_c}^{a(=1,2)}$ separates into a trace of Dirac gamma matrices and a trace of anomalous doubled isospin and colour degrees of freedom for the coset matrix (5.26,5.27); furthermore, we point out the trace $\mathfrak{tr}_{N_c}$ over the completely diagonal 'colour' terms with $r=1,\ldots,N_c=3$ of $\overline{f}_r(x_p)$. Since one also performs the trace over colour and isospin degrees of freedom, the integrated zero component of the helicity instanton number can only refer to field configurations of 'nucleons in its entity'; one can thereby classify the total number of nucleons in a field configuration composed of coset matrices $\hat{T}(x_p)$ by the derived helicity instanton number. As we act with the operator $\hat{\boldsymbol{\partial}}_{\boldsymbol{p,j}}$ of $\hat{\mathcal{B}}\;\hat{S}\;\hat{\Gamma}^j\;\hat{\boldsymbol{\partial}}_{\boldsymbol{p,j}}$ for creating the curl of $\hat{T}^{-1}\,\hat{\mathcal{B}}\,\hat{S}\,\hat{\Gamma}^k\,(\hat{\partial}_{p,k}\hat{T})$ in relation (5.20) in order to extract a Hopf invariant of $\Pi_3(\mathrm{SU}(2))=\mathsf{Z}$, one retains a single remaining combination and has also to mind the summation over the different fields $\overline{f}_r(x_p)$ following from the trace over completely diagonal colour degrees of freedom

$$\begin{aligned}
\int_{-\infty}^{+\infty}d^4x_p\;\hat{\partial}_{p,\mu}\Big(\psi^{\dagger}(x_p)\;\hat{\beta}\;\hat{\gamma}^{\mu}\;\hat{\gamma}_5\;\hat{\mathfrak{t}}_0\;\psi(x_p)\Big) &= \int d^3\vec{x}\;\psi^{\dagger}(x_p)\;\hat{\beta}\;\hat{\gamma}^0\;\hat{\gamma}_5\;\psi(x_p)\Big|_{x_p^0=-\infty}^{x_p^0=+\infty}= && (5.26)\\
&= \imath\left(-\frac{1}{2}\right)\int d^3\vec{x}\;\underset{N_f,\hat{\gamma}_{mn}^{(\mu)},Nc}{\overset{a(=1,2)}{\mathfrak{Tr}}}\Big[\hat{\mathcal{B}}\,\hat{\Gamma}^0\,\hat{\mathcal{B}}\,\hat{\Gamma}^i\,\hat{\Gamma}_5\,\hat{\mathcal{B}}\,\hat{\Gamma}^j\,\hat{\Gamma}_5\,\hat{\mathcal{B}}\,\hat{\Gamma}^k\,\hat{\Gamma}_5\;\times\\
&\times\;\hat{T}^{-1}(x_p)\;\hat{S}\;\big(\hat{\partial}_{p,i}\hat{T}(x_p)\big)\Big(\hat{S}\;\hat{\partial}_{p,j}\;\hat{T}^{-1}(x_p)\;\hat{S}\;\big(\hat{\partial}_{p,k}\hat{T}(x_p)\big)\Big)\Big]\Bigg|_{x_p^0=-\infty}^{x_p^0=+\infty}\;;
\end{aligned}$$

$$\underset{\hat{\gamma}_{mn}^{(\mu)}}{\mathfrak{tr}}\Big[\underbrace{\hat{\mathcal{B}}\,\hat{\Gamma}^0}_{-\imath\,\hat{1}}\;\underbrace{\hat{\mathcal{B}}\,\hat{\Gamma}^i\,\hat{\Gamma}_5}_{\imath\,\hat{\mathbb{S}}^i}\;\underbrace{\hat{\mathcal{B}}\,\hat{\Gamma}^j\,\hat{\Gamma}_5}_{\imath\,\hat{\mathbb{S}}^j}\;\underbrace{\hat{\mathcal{B}}\,\hat{\Gamma}^k\,\hat{\Gamma}_5}_{\imath\,\hat{\mathbb{S}}^k}\Big]=(-\imath)\;\imath^3\;4\;\imath\;\varepsilon^{ijk}\;\hat{1}^{ab}\;\hat{1}_{gf}\;\hat{1}_{sr}\;. \tag{5.27}$$

As we insert the relation (5.27) of the trace over anomalous doubled spin matrices $\hat{\mathbb{S}}^k$ (5.25) into (5.26), one introduces the three dimensional, completely anti-symmetric 'Levi-Civita symbol' 'ε^{ijk}' and attains Eq. (5.28) with remaining traces over isospin and colour degrees of freedom

$$\int_{-\infty}^{+\infty}d^4x_p\;\hat{\partial}_{p,\mu}\Big(\psi^{\dagger}(x_p)\;\hat{\beta}\;\hat{\gamma}^{\mu}\;\hat{\gamma}_5\;\hat{\mathfrak{t}}_0\;\psi(x_p)\Big)=\int d^3\vec{x}\;\psi^{\dagger}(x_p)\;\hat{\beta}\;\hat{\gamma}^0\;\hat{\gamma}_5\;\psi(x_p)\Big|_{x_p^0=-\infty}^{x_p^0=+\infty}= \tag{5.28}$$

$$= \;(-2)\int d^3\vec{x}\;\varepsilon^{ijk} \underset{N_f,N_c}{\overset{a(=1,2)}{\mathfrak{Tr}}}\Big[\hat{T}^{-1}(x_p)\;\hat{S}\;\big(\hat{\partial}_{p,i}\hat{T}(x_p)\,\big)\;\Big(\hat{S}\;\hat{\partial}_{p,j}\;\hat{T}^{-1}(x_p)\;\hat{S}\;\big(\hat{\partial}_{p,k}\hat{T}(x_p)\big)\Big)\Big]\;\Big|_{x_p^0=-\infty}^{x_p^0=+\infty}\;.$$

We apply the ansatz (5.21-5.24,5.29) for relation (5.28) and specify the complex fields $\overline{f}_r(x_p)$ in terms of their absolute value $|\overline{f}_r(x_p)\,|$ and phase $\phi_r(x_p)$

$$\begin{aligned} \hat{T}(x_p)\triangleq\hat{T}_0(x_p) &= \exp\left\{-\begin{pmatrix} 0 & \hat{X}_0(x_p) \\ \hat{X}_0^\dagger(x_p) & \end{pmatrix}\right\}\;;\quad \hat{X}_0(x_p)\triangleq\big(\hat{\tau}_2\big)_{gf}\;\overline{f}_r(x_p)\;\delta_{\overline{N};\overline{M}}\;; \\ \overline{f}_r(x_p) &= \big|\overline{f}_r(x_p)\big|\;\;\exp\big\{\imath\;\phi_r(x_p)\,\big\}\;, \end{aligned} \tag{5.29}$$

so that the transformed relation (5.28) of the original axial current conversation with chiral anomaly finally reduces to the 'curl of the gradient of the phase $\phi_r(x_p)$' and to the 'gradient of the hyperbolic sine of the absolute value $|\overline{f}_r(x_p)\,|$'

$$\begin{aligned} \frac{1}{2}\int_{-\infty}^{+\infty} d^4x_p\;\hat{\partial}_{p,\mu}\Big(\psi^\dagger(x_p)\;\hat{\beta}\;\hat{\gamma}^\mu\;\hat{\gamma}_5\;\hat{\mathfrak{t}}_0\;\psi(x_p)\Big) &= -\frac{\varepsilon^{\kappa\lambda\mu\nu}}{32\,\pi^2}\int d^4x_p\underset{N_f,N_c}{\mathfrak{tr}}\Big[\hat{\mathfrak{t}}_0\;\hat{F}_{\kappa\lambda}(x_p)\;\hat{F}_{\mu\nu}(x_p)\Big] = \\ = \;\imath\;(N_f=2)\sum_{r=1}^{N_c=3}\int d^3\vec{x}\;\varepsilon^{ijk}\;\Big(\hat{\partial}_{p,i}\sinh^4\big(\big|\overline{f}_r(x_p)\,\big|\,\big)\Big)&\;\Big(\hat{\partial}_{p,j}\hat{\partial}_{p,k}\phi_r(x_p)\Big)\;\Big|_{x_p^0=-\infty}^{x_p^0=+\infty}\;. \end{aligned} \tag{5.30}$$

It seems that the anti-symmetric combination $\varepsilon^{ijk}\;(\hat{\partial}_{p,j}\hat{\partial}_{p,k}\phi_r(x_p)\,)$ should result into completely vanishing terms; however, one has to take into account the multi-valued properties of phases $\phi_r(x_p)$ whose anti-symmetric, second order, spatial gradients commute everywhere except at line singularities which cause corresponding vortices. If one defines for the curl of the gradient of the phase $\phi_r(x_p)$ a line singularity (5.31) along the $x^3 = z$-axis in standard cylindrical coordinates $(\rho\,,\,\varphi\,,\,z)$ with locally orthonormal basis vectors $(\vec{e}_\rho\,,\,\vec{e}_\varphi\,,\,\vec{e}_z)$

$$\vec{\nabla}_{\vec{x}}\times\vec{\nabla}_{\vec{x}}\phi_r(x_p) = \vec{e}_z\;2\pi\;n_{z,r}\;\;\delta(x^1)\;\delta(x^2)\;;\quad (n_{z,r}\in\mathsf{Z})\;, \tag{5.31}$$

and applies appropriate boundary conditions (5.32) along the $x^3 = z$-axis, we can verify the quantization of relation (5.30) by '$n_{z,r}$' (5.34) for the BCS quark pair ansatz (5.29) of the coset matrix $\hat{T}_0(x_p)$. The phase $\phi_r(x_p)$ (5.33) is obtained by Stokes theorem from (5.31) according to the azimuthal symmetry within the cylinder coordinates

$$\Big(\sinh^2\big(\big|\overline{f}_r(t_p,x^3=0)\,\big|\big) = 0\quad;\quad \sinh^2\big(\big|\overline{f}_r(t_p,x^3=L_z)\,\big|\big) = \frac{1}{\sqrt{2\,\pi}}\Big)\;; \tag{5.32}$$

$$\begin{aligned} \vec{\nabla}_{\vec{x}}\times\vec{\nabla}_{\vec{x}}\phi_r(x_p) &= \vec{e}_z\;2\pi\;n_{z,r}\;\;\delta(x^1)\;\delta(x^2)\;; \\ (\text{Stokes theorem}\;\Longrightarrow)\;\;\vec{\nabla}_{\vec{x}}\phi_r(t_p,\vec{x}) &= \vec{e}_\varphi\;\frac{1}{\rho}\;\frac{\partial\phi_r(t_p,\vec{x})}{\partial\varphi} = \vec{e}_\varphi\;\frac{n_{z,r}}{\sqrt{(x^1)^2+(x^2)^2}}\;; \\ \phi_r(t_p,\vec{x}) &= n_{z,r}\;\;\varphi\;;\quad \varphi\in[0,2\pi)\;; \end{aligned} \tag{5.33}$$

$$\begin{aligned} \frac{1}{2}\int_{-\infty}^{+\infty} d^4x_p\;\hat{\partial}_{p,\mu}\Big(\psi^\dagger(x_p)\;\hat{\beta}\;\hat{\gamma}^\mu\;\hat{\gamma}_5\;\hat{\mathfrak{t}}_0\;\psi(x_p)\Big) &= -\frac{\varepsilon^{\kappa\lambda\mu\nu}}{32\,\pi^2}\int d^4x_p\underset{N_f,N_c}{\mathfrak{tr}}\Big[\hat{\mathfrak{t}}_0\;\hat{F}_{\kappa\lambda}(x_p)\;\hat{F}_{\mu\nu}(x_p)\Big] = \\ = \;\imath\;(N_f=2)\sum_{r=1}^{N_c=3}\int d^3\vec{x}\;\varepsilon^{ijk}\;\Big(\hat{\partial}_{p,i}\sinh^4\big(\big|\overline{f}_r(x_p)\,\big|\,\big)\Big)\;\Big(\hat{\partial}_{p,j}\hat{\partial}_{p,k}\phi_r(x_p)\Big)&\;\Big|_{x_p^0=-\infty}^{x_p^0=+\infty} = \imath(N_f=2)\sum_{r=1}^{N_c=3}n_{z,r}. \end{aligned} \tag{5.34}$$

Eqs. (5.30,5.34) can also be related to the Hopf invariant with one-form $\hat{\omega}_1(x_p)$ in a direct manner

$$\int\frac{\hat{\omega}_1(x_p)}{2\,\pi^2}\wedge\Big(d\hat{\omega}_1(x_p)\Big) = \text{integer number}\;, \tag{5.35}$$

by changing the one-forms $dx^i\hat{\partial}_{p,i}\sinh^4(|\overline{f}_r(x_p)\,|)$, $dx^k\hat{\partial}_{p,k}\phi_r(x_p)$ to corresponding one-forms $d\sinh^4(|\overline{f}_r(x_p)\,|)$, $d\phi_r(x_p)$ *which are not 'exact'*

$$\frac{1}{2}\int_{-\infty}^{+\infty} d^4x_p\;\hat{\partial}_{p,\mu}\Big(\psi^\dagger(x_p)\;\hat{\beta}\;\hat{\gamma}^\mu\;\hat{\gamma}_5\;\hat{\mathfrak{t}}_0\;\psi(x_p)\Big) = -\frac{\varepsilon^{\kappa\lambda\mu\nu}}{32\,\pi^2}\int d^4x_p\underset{N_f,N_c}{\mathfrak{tr}}\Big[\hat{\mathfrak{t}}_0\;\hat{F}_{\kappa\lambda}(x_p)\;\hat{F}_{\mu\nu}(x_p)\Big] = \tag{5.36}$$

$$
\begin{aligned}
&= \imath\,(N_f=2)\sum_{r=1}^{N_c=3}\int\Big(d\sinh^4\big(|\overline{f}_r(x_p)\,|\big)\Big)\;\wedge\;\Big(\underbrace{dx^j\,\hat{\partial}_{p,j}}_{\equiv d}\;d\phi_r(x_p)\Big)\Bigg|_{x_p^0=-\infty}^{x_p^0=+\infty}\;; \\
d &= d\rho\,\frac{\partial}{\partial\rho}+d\varphi\,\frac{\partial}{\partial\varphi}+dz\,\frac{\partial}{\partial z}\;.
\end{aligned}
$$

As one substitutes $d\phi_r(x_p)$, $d\sinh^4(|\overline{f}_r(x_p)\,|)$ according to following relations into (5.36)

$$
\begin{aligned}
d\phi_r(x_p) &= n_{z,r}\;\theta(\rho-\rho_z^{(0)})\;d\varphi+n_{\varphi,r}\;\theta(\rho_\varphi^{(0)}-\rho)\;dz\;2\,\pi/L_z\;; && (5.37)\\
d\sinh^4(|\overline{f}_r(x_p)\,|) &= \frac{1}{2\,\pi^2}\;\frac{d\phi_r(x_p)}{4}\;;\quad(0<z<L_z)\;; && (5.38)\\
\rho_\varphi^{(0)}>\rho_z^{(0)} &\quad;\quad n_{z,r}\;,\;n_{\varphi,r}\in\mathsf{Z}\;,
\end{aligned}
$$

one directly achieves a 'Hopf quantization' for $\Pi_3(S^2)=\mathsf{Z}$ from (5.36) with the integer numbers $n_{z,r}\;,\;n_{\varphi,r}$

$$
\begin{aligned}
&\frac{1}{2}\int_{-\infty}^{+\infty}d^4x_p\;\hat{\partial}_{p,\mu}\Big(\psi^\dagger(x_p)\;\hat{\beta}\;\hat{\gamma}^\mu\;\hat{\gamma}_5\;\hat{\mathfrak{t}}_0\;\psi(x_p)\Big)=-\frac{\varepsilon^{\kappa\lambda\mu\nu}}{32\,\pi^2}\int d^4x_p\underset{N_f,N_c}{\mathfrak{tr}}\Big[\hat{\mathfrak{t}}_0\;\hat{F}_{\kappa\lambda}(x_p)\;\hat{F}_{\mu\nu}(x_p)\Big]= \qquad (5.39)\\
&= \imath\,(N_f=2)\sum_{r=1}^{N_c=3}\int\Big(d\sinh^4\big(|\overline{f}_r(x_p)\,|\big)\Big)\;\wedge\;\Big(\underbrace{dx^j\,\hat{\partial}_{p,j}}_{\equiv d}\;d\phi_r(x_p)\Big)\Bigg|_{x_p^0=-\infty}^{x_p^0=+\infty}=\imath\,(N_f=2)\sum_{r=1}^{N_c=3}n_{z,r}\;,\;n_{\varphi,r}\;.
\end{aligned}
$$

Additonally, we note that the integrals (5.35,5.39) are invariant under deformations of the one-forms $\hat{\omega}_1(x_p)$, $d\phi_r(x_p)$, $d\sinh^4(|\overline{f}_r(x_p)\,|\,)$ (5.37,5.38) so that one has definitely determined a topological invariant, the 'Hopf invariant' [15].

6 Summary and conclusion

The rather involved appearance of section 3 contains the various HST's from the original QCD path integral with fermionic quark- and non-Abelian gauge fields to the corresponding self-energies. Since self-energies comprise the infinite sum of one-particle irreducible terms in a perturbation series, the path integral representation with self-energy matrices is advantageous to the original representation with fermionic matter- and non-Abelian gauge fields, especially due to various possible approximations. The total self-energy for the anomalous doubled dyadic product of quark fields consists of block diagonal density terms and off-diagonal BCS quark pairs which have been separated in a coset decomposition $\mathrm{SO}(N_0,N_0)\,/\,\mathrm{U}(N_0)\otimes\mathrm{U}(N_0)$ for a SSB with the unitary $\mathrm{U}(N_0)$ subgroup symmetry for the invariant ground or vacuum states ($N_0=(N_f=2)\times4_{\hat{\gamma}}\times(N_c=3)=24$). Bilinear observables follow from differentiating the prevailing form of the generating function with respect to the source field $\hat{\mathcal{J}}_{N;M}^{ba}(y_q,x_p)$ so that one can track the original observables of quark matter to their corresponding from in terms of the total self-energy. Since the path integral with self-energy generator $\mathrm{so}(N_0,N_0)$ can be separated by 'hinge' fields into purely density related parts $\mathrm{u}(N_0)$ and off-diagonal coset parts $\mathrm{so}(N_0,N_0)\,/\,\mathrm{u}(N_0)$ for BCS quark pairs, we can consider the density related part of the total path integral as background fields for the BCS terms. The density related path integral particularly allows for a mean field approximation of a composed gauge field $\mathcal{V}_\alpha^\mu(x_p)$ which replaces the original gauge fields $A_\alpha^\mu(x_p)$ and comprises colour dressed, scalar quark densities and self-energies for the original field strength tensor $\hat{F}_\alpha^{\mu\nu}(x_p)$. This mean field solution $\langle\hat{\mathcal{V}}(x_p)\rangle_{(3.59)}$ also has a non-hermitian part which has to comply with the correct sign of the original, imaginary '$-\imath\,\hat{\varepsilon}_p$' terms for a proper convergence of the generating function. Therefore, we accomplish a path integral which finally only contains coset elements $\mathrm{so}(N_0,N_0)\,/\,\mathrm{u}(N_0)$ propagating in density related, saddle point approximated gauge fields $\langle\hat{\mathcal{V}}(x_p)\rangle_{(3.59)}$. The remaining actions $\mathcal{A}_{DET}[\hat{T},\langle\hat{\mathcal{V}}\rangle_{(3.59)};\hat{\mathcal{J}}]$, $\mathcal{A}_{J_\psi}[\hat{T},\langle\hat{\mathcal{V}}\rangle_{(3.59)};\hat{\mathcal{J}}]$ consist of gradient operators '$\boldsymbol{\hat{\partial}_{p,\mu}}$' acting onto coset matrices which do not allow for simple, finite order gradient expansions; this particular problem is emphasized by a transformation to an 'interaction representation' so that the anomalous doubled one-particle operator $\hat{\mathcal{H}}_{\hat{\mathfrak{W}}}$ has only unsaturated gradient operators with spatially dependent Dirac gamma matrices. Since the actions $\mathcal{A}_{DET}[\hat{T}_{\hat{\mathfrak{W}}};\hat{\mathcal{J}}_{\hat{\mathfrak{W}}}]$ $\mathcal{A}_{J_{\psi;\hat{\mathfrak{W}}}}[\hat{T}_{\hat{\mathfrak{W}}};\hat{\mathcal{J}}_{\hat{\mathfrak{W}}}]$ mainly encompass the combination of operators $\hat{T}_{\hat{\mathfrak{W}}}^{-1}\;\hat{\mathcal{H}}_{\hat{\mathfrak{W}}}\;\hat{T}_{\hat{\mathfrak{W}}}\;\hat{\mathcal{H}}_{\hat{\mathfrak{W}}}^{\boldsymbol{-1}}$, a small momentum expansion for $\hat{T}_{\hat{\mathfrak{W}}}^{-1}\;\hat{\mathcal{H}}_{\hat{\mathfrak{W}}}\;\hat{T}_{\hat{\mathfrak{W}}}$ is always accompanied by a 'large' momentum expansion for $\hat{T}_{\hat{\mathfrak{W}}}\;\hat{\mathcal{H}}_{\hat{\mathfrak{W}}}^{\boldsymbol{-1}}\;\hat{T}_{\hat{\mathfrak{W}}}^{-1}$ due to the inverse operator $\hat{\mathcal{H}}_{\hat{\mathfrak{W}}}^{\boldsymbol{-1}}$ of pure gradients in

the trace relations. We suggest the specific integral representations (4.27-4.29) of the logarithm and of the inverse of an operator (4.29) which can simplify the computation of observables.

It is of peculiar interest to investigate the path integral with coset matrices $\hat{T}(x_p)$ and approximated background fields $\langle\hat{\mathcal{V}}(x_p)\rangle_{(3.59)}$ for nontrivial topologies especially in comparison to the original Skyrme model with homotopy group $\Pi_3(\mathrm{SU}(2)\,)=\mathsf{Z}$. As one restricts to the anti-symmetric, quaternion-valued complex eigenvalues of coset generators $\hat{X}(x_p)$, $\hat{X}^\dagger(x_p)$ with Pauli matrix $(\hat{\tau}_2)_{gf}$, one only has two real angle degrees of freedom instead of the corresponding three within the Skyrme model. Therefore, the Hopf mapping $\Pi_3(S^2)=\mathsf{Z}$ with the Hopf invariant only remains for nontrivial field configurations of the quaternionic, anti-symmetric complex eigenvalues. In fact, we can extract a Hopf invariant from the axial current conversation with the chiral anomaly in the massless limit and can determine nontrivial field configurations of the complex eigenvalue angles. This is briefly exemplified in cylindrical geometry and can also be illustrated in toroidal coordinates. The nontrivial Hopf mappings of $\Pi_3(S^2)=\mathsf{Z}$ can be intuitively understood as one considers the 'preimage' of the S^2 sphere where every point of the S^2 sphere corresponds to a S^1 loop in the spatial S^3 sphere. Since there occurs an imaginary factor with the extracted Hopf invariant, the nontrivial field configurations correspond to 'helicity instantons' following from the axial current conversation with the chiral anomaly. These 'helicity instantons' thereby classify the prevailing field configuration of 'nucleons' according to the summations over isospin- and colour-degrees of freedom. For that reason, our derived path integral with coset matrices $\hat{T}(x_p)$ is more closely related to the Skyrme-Faddeev model with $\Pi_3(S^2)=\mathsf{Z}$ than to the original Skyrme model $\Pi_3(\mathrm{SU}(2)\,)=\mathsf{Z}$ which is regarded as an effective field theory for QCD in the limit of infinite colour degrees of freedom $N_c\rightarrow\infty$.

A Hilbert space of anomalous doubled operators and their representations

According to chapter 4 of Ref. [11], we describe how the gradient operators $\hat{\boldsymbol{\partial}}_{\boldsymbol{p,\mu}}$ act on the coset matrices $\hat{T}(x_p)$ in the operator $\hat{\mathcal{O}}_{N;M}^{ba}(y_q,x_p)$ (3.111) with $\Delta\hat{\mathcal{H}}(x_p)$ and on densities as e.g. the quark self-energy density $\hat{\sigma}_D^{(\alpha;\kappa)}(x_p)$. One can consider some representation as the 3+1 dimensional spacetime-coordinates where the operators $\hat{T}$, $\hat{Q}$, $\delta\hat{\lambda}$, $\hat{\sigma}_D^{(\alpha;\kappa)}$ are defined to be diagonal in the spacetime variables and are given by the matrix fields $\hat{T}_{M;N}^{ab}(x_p)$, $\hat{Q}_{M;N}^{aa}(x_p)$ and the scalar density fields $\delta\hat{\lambda}_M(x_p)$, $\sigma_D^{(\alpha;\kappa)}(x_p)$. However, it has to be taken into account that the square root of the determinant follows from integration over the bilinear anti-commuting fields which are doubled by their complex conjugates $\psi_M^*(x_p)$. Consequently a Hilbert space for $\psi_M(x_p)$ with 'ket' $|\psi_M\rangle$ has also to be doubled by its 'dual' space $\overline{|\psi_M\rangle}=\langle\psi_M|$ the 'bra'. The unsaturated operators $\hat{\boldsymbol{\partial}}_{\boldsymbol{p,\mu}}$ are printed in boldface in order to distinguish from the matrix functions as $(\hat{\partial}_{p,\mu}\hat{T}(x_p))$ embraced in brackets, denoting the limited action of the derivative on the prevailing coset matrix $\hat{T}(x_p)$. These 'saturated' gradient operators are not involved in further derivative actions on matrices or fields outside the parentheses and are therefore not printed in bold type.

The detailed structure of the Hilbert space with its doubled dual part $\langle\psi_M|=\overline{|\psi_M\rangle}$ is important because the doubled operator $\hat{\mathcal{H}}(x_p)$ (3.87,3.88) applies the transpose $\hat{H}^T(x_p)$ in the '22' block instead of $\hat{H}(x_p)$ as in the '11' part. An operator in quantum mechanics is defined by the mapping and the space on which it acts. Completely different results can follow if one considers for one and the same mapping of an operator different spaces where the operator transforms the prevailing states. The path integral (3.110-3.116) follows by integration over the doubled anti-commuting fields $\Psi_M^a(x_p)$ from (3.103). The corresponding doubled abstract states $\widehat{|\psi_M\rangle}^{a(=1,2)}$ with internal space label $M,\ N,\ M',\ N',\ldots$ are defined in (A.1)

$$\Psi_M^{a(=1,2)}(x_p)=\begin{pmatrix}\psi_M(x_p)\\ \psi_M^*(x_p)\end{pmatrix}\propto\widehat{|\Psi_M\rangle}^{a(=1,2)}=\begin{pmatrix}|\psi_M\rangle^{a=1}\\ \overline{|\psi_M\rangle}^{a=2}\end{pmatrix}. \tag{A.1}$$

The appropriate abstract Hilbert space has to be introduced for the definition of the operators $\hat{\boldsymbol{\partial}}_{\boldsymbol{p,\mu}}$ in the determinant-action $\langle\mathcal{A}_{DET}[\hat{T},\hat{\mathcal{V}};\hat{\mathcal{J}}]\rangle_{\hat{\mathcal{V}}}$ and $\langle\mathcal{A}_{J_\psi}[\hat{T},\hat{\mathcal{V}};\hat{\mathcal{J}}]\rangle_{\hat{\mathcal{V}}}$ (3.114,3.115). According to the doubling with the dual part $\overline{|\psi_M\rangle}=\langle\psi_M|$, we have an anti-linear property in the second part $\widehat{|\psi_M\rangle}^{a=2}$

$$\widehat{|c\ \Psi_M\rangle}=\begin{pmatrix}c\ |\psi_M\rangle\\ c^*\ \overline{|\psi_M\rangle}\end{pmatrix}\quad;\qquad c\in\mathsf{C}\quad. \tag{A.2}$$

Furthermore, we simultaneously have the unitary and 'anti'-unitary representation of $U(N_0)$ $(N_0=N_f\times4_{\hat{\gamma}}\times N_c)$ in the '11' and '22' block, respectively. This is in accordance with a theorem of Wigner that a symmetry in quantum mechanics can have a unitary or anti-unitary realization [20]. The corresponding Hilbert space for 3+1 dimensional spacetime has therefore also to be doubled with the anti-linear part

$$\widehat{|x_p\rangle}^{a(=1,2)}=\begin{pmatrix}|x_p\rangle^{a=1}\\ \overline{|x_p\rangle}^{a=2}\end{pmatrix}=\begin{pmatrix}|x_p\rangle\\ \langle x_p|\end{pmatrix}\ ; \tag{A.3}$$

$$\begin{array}{lllllllll}
\langle x_p^0|y_q^0\rangle &=& \delta_{pq}\ \delta_{x_p^0,y_q^0}\ ; & \langle\vec{x}|\vec{y}\rangle &=& \delta_{\vec{x},\vec{y}}\ ; & \langle x_p|y_q\rangle &=& \delta_{pq}\ \delta_{x_p^0,y_q^0}\ \delta_{\vec{x},\vec{y}}\ ;\\
\langle k_p^0|p_q^0\rangle &=& \delta_{p,q}\ \delta_{k_p^0,p_q^0}\ ; & \langle\vec{k}|\vec{p}\rangle &=& \delta_{\vec{k},\vec{p}}\ ; & \langle k_p|p_q\rangle &=& \delta_{pq}\ \delta_{k_p^0,p_q^0}\ \delta_{\vec{k},\vec{p}}\ ;\\
\langle x_p^0|k_q^0\rangle &=& \delta_{pq}\ \exp\{-\imath\ k_p^0\cdot x_p^0\}\ ; & \langle\vec{x}|\vec{k}\rangle &=& \exp\{\imath\ \vec{k}\cdot\vec{x}\}\ ; & \langle x_p|k_q\rangle &=& \delta_{pq}\ \exp\{\imath(\vec{k}\cdot\vec{x}-k_p^0\cdot x_p^0)\}\ ;
\end{array} \tag{A.4}$$

$$\sum_{a=1,2}{}^{a}\widehat{\langle x_p|}\widehat{k_q\rangle}^{a} = \langle x_p|k_q\rangle+\overline{\langle x_p|}\overline{k_q\rangle}\ ; \tag{A.5}$$

$${}^{a=1}\widehat{\langle x_p|}\widehat{k_q\rangle}^{a=1} = \langle x_p|k_q\rangle=\delta_{pq}\ \exp\{\imath(\vec{k}\cdot\vec{x}-k_p^0\cdot x_p^0)\}\ ; \tag{A.6}$$

$${}^{a=2}\widehat{\langle x_p|}\widehat{k_q\rangle}^{a=2} = \overline{\langle x_p|}\overline{k_q\rangle}=\langle k_q|x_p\rangle=\big(\langle x_p|k_q\rangle\big)^*=\delta_{pq}\ \exp\{-\imath(\vec{k}\cdot\vec{x}-k_p^0\cdot x_p^0)\}\ . \tag{A.7}$$

The total unit operators with the unitary and anti-unitary parts are listed in Eqs. (A.8,A.9) for 3+1 dimensional spacetime and momentum-energy states. We have to combine the contour integrals of forward and backward propagation with the contour metric $\eta_{p=\pm}=\pm$ (2.15,2.16) so that the defining relations (A.3-A.7) exactly match with the properties of the unit operators (A.8,A.9)

$$\begin{aligned}
\hat{1} &= \begin{pmatrix} \hat{1}^{11} & \\ & \hat{1}^{22} \end{pmatrix} = \sum_{p=\pm}\int_{-\infty}^{+\infty} d^4x_p\,\mathcal{N}\begin{pmatrix} |x_p\rangle\langle x_p| & \\ & \overline{|x_p\rangle}\,\overline{\langle x_p|} \end{pmatrix} && \text{(A.8)}\\
&= \int_C d^4x_p\,\eta_p\,\mathcal{N}\begin{pmatrix} |x_p\rangle\langle x_p| & \\ & \overline{|x_p\rangle}\,\overline{\langle x_p|} \end{pmatrix} = \int_C d^4x_p\,\eta_p\,\mathcal{N}\sum_{a=1,2}\widehat{|x_p\rangle}^{a(=1,2)}\ {}^{a(=1,2)}\widehat{\langle x_p|}\ ;\quad \mathcal{N}=\frac{1}{(\Delta x)^4}\ ;\\
\hat{1} &= \begin{pmatrix} \hat{1}^{11} & \\ & \hat{1}^{22} \end{pmatrix} = \sum_{p=\pm}\int_{-\infty}^{+\infty} d^4k_p\,\mathcal{N}_k\begin{pmatrix} |k_p\rangle\langle k_p| & \\ & \overline{|k_p\rangle}\,\overline{\langle k_p|} \end{pmatrix} && \text{(A.9)}\\
&= \int_C d^4k_p\,\eta_p\,\mathcal{N}_k\begin{pmatrix} |k_p\rangle\langle k_p| & \\ & \overline{|k_p\rangle}\,\overline{\langle k_p|} \end{pmatrix} = \int_C d^4k_p\,\eta_p\,\mathcal{N}_k\sum_{a=1,2}\widehat{|k_p\rangle}^{a(=1,2)}\ {}^{a(=1,2)}\widehat{\langle k_p|}\ ;\quad \mathcal{N}_k=\frac{1}{(\Delta k)^4}\ .
\end{aligned}$$

The 3+1 spacetime '$x_p=\{x_p^0,\vec{x}\}$' and four-momentum '$k_p=\{k_p^0,\vec{k}\}$' representations of the abstract doubled Hilbert space states (A.2) are given in (A.10,A.11) where the relations (A.3-A.7) are applied, including the anti-unitary second part

$$\begin{aligned}
{}^{a}\widehat{\langle x_p|\Psi_M\rangle}^{a} &= \begin{pmatrix} \langle x_p|\psi_M\rangle \\ \overline{\langle x_p|}\overline{|\psi_M\rangle} \end{pmatrix}^{a(=1,2)} = \begin{pmatrix} \psi_M(x_p) \\ \langle\psi_M|x_p\rangle \end{pmatrix}^{a(=1,2)} = \begin{pmatrix} \psi_M(x_p) \\ \psi_M^*(x_p) \end{pmatrix}^{a(=1,2)}\ ; && \text{(A.10)}\\
{}^{a}\widehat{\langle k_p|\Psi_M\rangle}^{a} &= \begin{pmatrix} \langle k_p|\psi_M\rangle \\ \overline{\langle k_p|}\overline{|\psi_M\rangle} \end{pmatrix}^{a(=1/2)} = \begin{pmatrix} \psi_M(k_p) \\ \langle\psi_M|k_p\rangle \end{pmatrix}^{a(=1,2)} = \begin{pmatrix} \psi_M(k_p) \\ \psi_M^*(k_p) \end{pmatrix}^{a(=1,2)}\ . && \text{(A.11)}
\end{aligned}$$

The scalar self-energy densities $\hat{\sigma}_D^{(\alpha;\kappa)}$ operate on the doubled spacetime state $\widehat{|y_q\rangle}^{a}$ as in the well-known case of an annihilation operator on coherent states, but one has to incorporate the contour metric η_q and has to consider that the resulting coherent state field $\sigma_D^{(\alpha;\kappa)}(y_q)$ only takes real values (A.12). This additional contour metric η_p has to be taken into account for the one-particle operator $\hat{\mathcal{H}}_{N;M}^{ba}(x_p)$, the self-energy density $\hat{\sigma}_D^{(\alpha;\kappa)}(x_p)$ and $\delta\hat{\Sigma}_{M;N}^{ab}(x_p)$ because these operators appear in the original path integrals and only lead to diagonal matrix elements in the time contour due to a missing disorder. An ensemble average with a random potential would include non-diagonal terms in the total self-energy concerning the time contour metric [12]. However, the abstract operator action with the anomalous parts $\hat{Y}_{M;N}^{ab}(x_p)$ in $\hat{T}_{M;N}^{ab}(x_p)$ does not involve an additional time contour metric in the considered case without disorder

$$\begin{aligned}
\hat{\sigma}_D^{(\alpha;\kappa)}\ \widehat{|y_q\rangle}^{a} &= \eta_q\,\sigma_D^{(\alpha;\kappa)}(y_q)\,\widehat{|y_q\rangle}^{a}\ ;\qquad \sigma_D^{(\alpha;\kappa)}(y_q)\in\mathsf{R}\ ; && \text{(A.12)}\\
\hat{T}_{M;N}^{ab}\ \widehat{|y_q\rangle}^{b} &= \hat{T}_{M;N}^{ab}(y_q)\,\widehat{|y_q\rangle}^{b}\ . && \text{(A.13)}
\end{aligned}$$

Using the definitions (A.3-A.7) of the abstract doubled Hilbert space, we can pursue the various steps for calculating matrix elements ${}^{a}\widehat{\langle x_p|\hat{T}_{M;N}^{ab}|y_q\rangle}^{b}$ from the generating operators $\hat{Y}_{M;N}^{ab}$, $\hat{X}_{M;N}$, $\hat{X}_{M;N}^{\dagger}$ in the exponential of $\hat{T}$. However, the operator $\hat{X}_{M;N}$ of the anomalous parts is constructed from two field operators $\hat{\psi}_M\ \hat{\psi}_N$ so that matrix elements $\langle x_p|\hat{X}_{M;N}\overline{|y_q\rangle}$ (A.15,A.16) result in the expansion of $\hat{T}_{M;N}^{ab}$ (A.14), combining also Hilbert states with linear and anti-linear parts

$$\begin{aligned}
{}^{a}\widehat{\langle x_p|\hat{T}_{M;N}^{ab}|y_q\rangle}^{b} &= {}^{a}\widehat{\langle x_p|}\big(\hat{1}-\hat{Y}_{M;N}^{ab}\pm\ldots\big)\widehat{|y_q\rangle}^{b} = && \text{(A.14)}\\
&= \delta_{ab}\,\delta_{M;N}\,\delta_{\vec{x},\vec{y}}\,\delta_{pq}\,\delta_{x_p^0,y_q^0}-\begin{pmatrix} \langle x_p| \\ \overline{\langle x_p|} \end{pmatrix}^{T}\begin{pmatrix} 0 & \hat{X}_{M;N} \\ \hat{X}_{M;N}^{\dagger} & 0 \end{pmatrix}\begin{pmatrix} |y_q\rangle \\ \overline{|y_q\rangle} \end{pmatrix}\pm\ldots
\end{aligned}$$

$$= \delta_{\vec{x},\vec{y}}\;\delta_{pq}\;\delta_{x_p^0,y_q^0}\;\left[\hat{1}\;\delta_{ab}\;\delta_{M;N}-\begin{pmatrix} 0 & \hat{X}_{M;N}(x_p) \\ \hat{X}^{\dagger}_{M;N}(x_p) & 0\end{pmatrix}^{ab}\pm\ldots\right]\;;$$

$$\begin{aligned}
\langle x_p|\hat{X}_{M;N}|\overline{y_q\rangle} &= \hat{X}_{M;N}(x_p)\;\delta_{\vec{x},\vec{y}}\;\delta_{pq}\;\delta_{x_p^0,y_q^0}\;; && \text{(A.15)}\\
\overline{\langle x_p}|\hat{X}^{\dagger}_{M;N}|y_q\rangle &= \hat{X}^{\dagger}_{M;N}(x_p)\;\delta_{\vec{x},\vec{y}}\;\delta_{pq}\;\delta_{x_p^0,y_q^0}\;; && \text{(A.16)}\\
\Big(\langle x_p|\hat{X}_{M;N}|\overline{y_q\rangle}\Big)^{*} &= \overline{\langle y_q}|\hat{X}^{\dagger}_{M;N}|x_p\rangle=\overline{\langle x_p}|\hat{X}^{\dagger}_{M;N}|y_q\rangle\;; && \text{(A.17)}\\
\hat{X}_{M;N}(x_p) &\propto \big(\psi_M(x_p)\;\psi_N(x_p)\big)\;; && \text{(A.18)}\\
\hat{X}^{\dagger}_{M;N}(x_p) &\propto \big(\psi_M(x_p)\;\psi_N(x_p)\big)^{\dagger}=\big(\psi^{*}_N(x_p)\;\psi^{*}_M(x_p)\big)^{T}= && \text{(A.19)}\\
&= \big(\psi^{*}_M(x_p)\;\psi^{*}_N(x_p)\big)\;;\qquad \psi_M(x_p)\;,\;\psi_N(x_p)\;\in\mathsf{C}_{odd}\;.
\end{aligned}$$

Summarizing the effect of the doubling of Hilbert space states with the anti-unitary extension, we list the matrix elements of the density parts (A.20,A.21), always containing the time contour metric η_p, and the pair condensates (A.22), as derived with the properties (A.15-A.19) in the expansion (A.14)

$$\begin{aligned}
{}^{a}\widehat{\langle x_p}|\hat{\mathcal{H}}|\widehat{y_q\rangle}^{b} &= \delta_{ab}\;\delta_{\vec{x},\vec{y}}\;\eta_p\;\delta_{pq}\;\delta_{x_p^0,y_q^0}\begin{pmatrix}\hat{H}_{M;N}(x_p) & 0\\ 0 & \hat{H}^{T}_{M;N}(x_p)\end{pmatrix}^{ab}\;; && \text{(A.20)}\\
{}^{a}\widehat{\langle x_p}|\hat{\sigma}_D^{(\alpha;\kappa)}\;\hat{1}_{2N_0\times 2N_0}+\delta\hat{\Sigma}^{ab}_{M;N}|\widehat{y_q\rangle}^{b} &= \delta_{\vec{x},\vec{y}}\;\eta_p\;\delta_{pq}\;\delta_{x_p^0,y_q^0}\;\Big(\sigma_D^{(\alpha;\kappa)}(x_p)\big)\;\delta_{ab}\;\delta_{M;N}+\delta\hat{\Sigma}^{ab}_{M;N}(x_p)\Big)\;; && \text{(A.21)}\\
{}^{a}\widehat{\langle x_p}|\hat{T}^{ab}_{M;N}|\widehat{y_q\rangle}^{b} &= \hat{T}^{ab}_{M;N}(x_p)\;\;\delta_{\vec{x},\vec{y}}\;\delta_{pq}\;\delta_{x_p^0,y_q^0}\;. && \text{(A.22)}
\end{aligned}$$

The source field $J^{a}_{\psi;M}(x_p)$ for the BEC wave function and the source matrix $\widetilde{\mathfrak{J}}^{ab}_{M;N}$ for generating observables are defined in (A.23,A.24) for corresponding doubled states and matrices

$$\begin{aligned}
J^{a}_{\psi;M}(x_p) &= {}^{a}\widehat{\langle x_p}|\widehat{J_{\psi;M}\rangle}^{a}=\begin{pmatrix} j_{\psi;M}(x_p)\\ j^{*}_{\psi;M}(x_p)\end{pmatrix}^{a}\;; && \text{(A.23)}\\
\widetilde{\mathfrak{J}}^{ab}_{M;N}(x_p,y_q) &= {}^{a}\widehat{\langle x_p}|\widetilde{\mathfrak{J}}^{ab}_{M;N}|\widehat{y_q\rangle}^{b}=\hat{I}\;\hat{S}\;\eta_p\;\frac{\hat{\mathfrak{J}}^{ab}_{M;N}(x_p,y_q)}{\mathcal{N}}\;\eta_q\;\hat{S}\;\hat{I}\quad. && \text{(A.24)}
\end{aligned}$$

The definition of the unit operators (A.8,A.9) with time contour integration *and* additional contour metric can be transformed to a trace relation as one can change the unit operator $\hat{1}=\sum_n|n\rangle\langle n|$ of a complete set of states $|n\rangle$ in ordinary quantum mechanics to a trace relation $\mathrm{tr}[\ldots]=\sum_n\langle n|\ldots|n\rangle$. However, we have to distinguish between the anomalous doubled trace $\mathrm{TR}^{a(=1,2)}_{\int_C d^4x_p\;\eta_p}[\ldots]$ (A.25) with the anti-unitary second part $\overline{\langle x_p}|\ldots|\overline{x_p\rangle}$ and the ordinary trace $\mathrm{tr}_{\int_C d^4x_p\;\eta_p}[\ldots]$ (A.26) also with time contour integration, but without the anomalous doubled anti-unitary part

$$\begin{aligned}
\underset{\int_C d^4x_p\;\eta_p}{\overset{a(=1,2)}{\mathrm{TR}}}\Big[\ldots\Big] &:= \sum_{p=\pm}\int_{-\infty}^{+\infty}d^4x_p\;\mathcal{N}\sum_{a=1,2}{}^{a}\widehat{\langle x_p}|\ldots|\widehat{x_p\rangle}^{a}=\int_C d^4x_p\;\eta_p\;\mathcal{N}\sum_{a=1,2}{}^{a}\widehat{\langle x_p}|\ldots|\widehat{x_p\rangle}^{a} && \text{(A.25)}\\
&= \int_{-\infty}^{+\infty}dx_+^0\int_{L^3}d^3\vec{x}\;\mathcal{N}\sum_{a=1,2}{}^{a}\widehat{\langle x_+}|\ldots|\widehat{x_+\rangle}^{a}+\int_{-\infty}^{+\infty}dx_-^0\int_{L^3}d^3\vec{x}\;\mathcal{N}\sum_{a=1,2}{}^{a}\widehat{\langle x_-}|\ldots|\widehat{x_-\rangle}^{a}\;;\\
\underset{\int_C d^4x_p\;\eta_p}{\mathrm{tr}}\Big[\ldots\Big] &:= \sum_{p=\pm}\int_{-\infty}^{+\infty}d^4x_p\;\mathcal{N}\;\langle x_p|\ldots|x_p\rangle=\int_C d^4x_p\;\eta_p\;\mathcal{N}\;\langle x_p|\ldots|x_p\rangle && \text{(A.26)}\\
&= \int_{-\infty}^{+\infty}dx_+^0\int_{L^3}d^3\vec{x}\;\mathcal{N}\;\langle x_+|\ldots|x_+\rangle+\int_{-\infty}^{+\infty}dx_-^0\int_{L^3}d^3\vec{x}\;\mathcal{N}\;\langle x_-|\ldots|x_-\rangle\;.
\end{aligned}$$

In a similar manner one introduces the traces of the four-momentum '$k_p = \{k_p^0, \vec{k}\}$' which are needed for a lowest order momentum and gradient expansion of the action for the determinant

$$\underset{\int_C d^4k_p\;\eta_p}{\overset{a(=1,2)}{\mathrm{TR}}}\Big[\ldots\Big] := \sum_{p=\pm}\int_{-\infty}^{+\infty} d^4k_p\;\mathcal{N}_k \sum_{a=1,2} {}^a\widehat{\langle k_p}|\ldots|\widehat{k_p\rangle}^a = \int_C d^4k_p\;\eta_p\;\mathcal{N}_k \sum_{a=1,2} {}^a\widehat{\langle k_p}|\ldots|\widehat{k_p\rangle}^a \tag{A.27}$$
$$= \int_{-\infty}^{+\infty} dk_+^0 \int d^3\vec{k}\;\mathcal{N}_k \sum_{a=1,2} {}^a\widehat{\langle k_+}|\ldots|\widehat{k_+\rangle}^a + \int_{-\infty}^{+\infty} dk_-^0 \int d^3\vec{k}\;\mathcal{N}_k \sum_{a=1,2} {}^a\widehat{\langle k_-}|\ldots|\widehat{k_-\rangle}^a\;;$$

$$\underset{\int_C d^4k_p\;\eta_p}{\mathrm{tr}}\Big[\ldots\Big] := \sum_{p=\pm}\int_{-\infty}^{+\infty} d^4k_p\;\mathcal{N}_k\;\langle k_p|\ldots|k_p\rangle = \int_C d^4k_p\;\eta_p\;\mathcal{N}_k\;\langle k_p|\ldots|k_p\rangle \tag{A.28}$$
$$= \int_{-\infty}^{+\infty} dk_+^0 \int d^3\vec{k}\;\mathcal{N}_k\;\langle k_+|\ldots|k_+\rangle + \int_{-\infty}^{+\infty} dk_-^0 \int d^3\vec{k}\;\mathcal{N}_k\;\langle k_-|\ldots|k_-\rangle\;.$$

B Ward identities corresponding to the gauge symmetries of the coset space

B.1 Ward identities and gauge invariance of the background potential

As we examine the path integral (3.59) and the separated actions $\mathcal{A}_{DET}[\hat{T},\hat{\mathcal{V}};\hat{\mathcal{J}}]$, $\mathcal{A}_{J_\psi}[\hat{T},\hat{\mathcal{V}};\hat{\mathcal{J}}]$ (3.110-3.116), we recognize the common background potential $\mathcal{V}_\alpha^\mu(x_p)$ (3.60) in $\hat{H}_{N;M}(y_q,x_p)$ (3.88) or in its anomalous doubled version $\hat{\mathcal{H}}_{N;M}^{ba}(y_q,x_p)$ (3.87) with the transpose $\hat{H}_{N;M}^T(y_q,x_p)$ in the '22' block section. The composed background potential $\mathcal{V}_\alpha^\mu(x_p)$ substitutes the original gauge fields $A_\alpha^\mu(x_p)$ of $\mathrm{SU}_c(N_c=3)$ in axial gauge $A_\alpha^\nu(x_p)\;n_\nu = 0$ ($\vec{n}\cdot\vec{n}=1$) which is obtained by the standard exponential integral representation of the delta functions $\delta(\,A_\alpha^\nu(x_p)\;n_\nu\,)$ with auxiliary real fields $s_\alpha(x_p)$. In consequence, the auxiliary integrals of $s_\alpha(x_p)$ for the delta functions $\delta(\,A_\alpha^\nu(x_p)\;n_\nu\,)$ fix the axial gauge so that the original path integrals avoid the multiple (in fact infinite) weighting of physically equivalent configurations which just differ by a gauge transformation. Despite of this original axial gauge fixing by $s_\alpha(x_p)$, a gauge invariance is still retained for the composed background gauge field $\mathcal{V}_\alpha^\mu(x_p)$ (3.60).

The background gauge field $\mathcal{V}_\alpha^\mu(x_p)$ (B.1) consists of the self-energy field strength tensor $\hat{\mathfrak{S}}_\alpha^{(\hat{F})\mu\nu}(x_p)$, its inverse with $\mathrm{SU}_c(N_c=3)$ structure constants $C_{\alpha\beta\gamma}$, its derivative and additionally of the self-energy quark densities $\sigma_D^{(\alpha;\kappa)}(x_p)$ which are dressed by the eigenvectors $\hat{\mathfrak{B}}_{\hat{F};\beta(\alpha)}^{\mu(\kappa)}(x_p)$ of the self-energy field strength term $C_{\alpha\beta\gamma}\;\hat{\mathfrak{S}}_{\alpha;\mu\nu}^{(\hat{F})}(x_p)$

$$\mathcal{V}_\beta^\mu(x_p) = \Big[\Big(\hat{\partial}_p^\lambda\hat{\mathfrak{S}}_{\gamma;\nu\lambda}^{(\hat{F})}(x_p)\Big) - s_\gamma(x_p)\;n_\nu\Big]\Big[-\imath\,\hat{\mathfrak{e}}_p^{(\hat{F})} + C_{\alpha\beta'\gamma'}\;\hat{\mathfrak{S}}_\alpha^{(\hat{F})\mu'\nu'}(x_p)\Big]_{\gamma\beta}^{-1;\nu\mu} + \tag{B.1}$$
$$+ \frac{1}{2}\sum_{(\alpha)=1,.,8}^{(\kappa)=0,.,3}\hat{\mathfrak{B}}_{\hat{F};\beta(\alpha)}^{\mu(\kappa)}(x_p)\;\;\sigma_D^{(\alpha;\kappa)}(x_p)\;;$$

$$C_{\alpha\beta\gamma}\;\hat{\mathfrak{S}}_\alpha^{(\hat{F})\mu\nu}(x_p) := \sum_{(\alpha)=1,.,8}^{(\kappa)=0,.,3}\hat{\mathfrak{B}}_{\hat{F};\beta(\alpha)}^{\mu(\kappa)}(x_p)\;\hat{\mathfrak{b}}_{(\alpha;\kappa)}^{(\hat{F})}(x_p)\;\hat{\mathfrak{B}}_{\hat{F};(\alpha)\gamma}^{T,(\kappa)\nu}(x_p)\;; \tag{B.2}$$

$$d[\hat{\mathfrak{S}}_\alpha^{(\hat{F})\mu\nu}(x_p)] \to d[\hat{\mathfrak{S}}_\alpha^{(\hat{F})\mu\nu}(x_p);\hat{\mathfrak{B}}_{\hat{F};\beta(\alpha)}^{\mu(\kappa)}(x_p);\hat{\mathfrak{b}}_{(\alpha;\kappa)}^{(\hat{F})}(x_p)]\;; \tag{B.3}$$

$$d[\hat{\mathfrak{S}}_\alpha^{(\hat{F})\mu\nu}(x_p);\hat{\mathfrak{B}}_{\hat{F};\beta(\alpha)}^{\mu(\kappa)}(x_p);\hat{\mathfrak{b}}_{(\alpha;\kappa)}^{(\hat{F})}(x_p)] = d[\hat{\mathfrak{S}}_\alpha^{(\hat{F})\mu\nu}(x_p)]\;\;d[\hat{\mathfrak{B}}_{\hat{F};\beta(\alpha)}^{\mu(\kappa)}(x_p);\hat{\mathfrak{b}}_{(\alpha;\kappa)}^{(\hat{F})}(x_p)]\times \tag{B.4}$$
$$\times\;\;\Bigg\{\prod_{\{x_p\}}\delta\Bigg(C_{\alpha\beta\gamma}\;\hat{\mathfrak{S}}_\alpha^{(\hat{F})\mu\nu}(x_p) - \sum_{(\alpha)=1,.,8}^{(\kappa)=0,.,3}\hat{\mathfrak{B}}_{\hat{F};\beta(\alpha)}^{\mu(\kappa)}(x_p)\;\hat{\mathfrak{b}}_{(\alpha;\kappa)}^{(\hat{F})}(x_p)\;\hat{\mathfrak{B}}_{\hat{F};(\alpha)\gamma}^{T,(\kappa)\nu}(x_p)\Bigg)\Bigg\}\;.$$

The anomalous doubled one-particle operator $\hat{\mathcal{H}}(x_p)$ contains the composed gauge fields $\mathcal{V}_\alpha^\mu(x_p)$, $\hat{\mathcal{V}}(x_p)$ or more precisely the anomalous doubled version $\hat{\mathcal{V}}^\mu(x_p) = \hat{\mathfrak{T}}_\alpha\;\mathcal{V}_\alpha^\mu(x_p)$, $\hat{\mathcal{V}}(x_p)$ with $\mathrm{SU}_c(N_c=3)$ generators $\hat{\mathfrak{T}}_\alpha$ which are doubled by the

transpose $-\hat{t}_\alpha^T$ in the '22' block

$$
\begin{aligned}
\hat{\mathfrak{H}}^{ba}_{N;M}(y_q,x_p) &= \delta^{(4)}(y_q-x_p)\;\eta_q\;\delta_{qp}\begin{pmatrix}\hat{H}_{N;M}(x_p) & \\ & \hat{H}^T_{N;M}(x_p)\end{pmatrix}^{ba} ; && \text{(B.5)}\\
\hat{H}(x_p) &= \big[\hat{\beta}(\hat{\partial\!\!\!/}_p+\imath\,\hat{\mathcal{V}\!\!\!/}(x_p)-\imath\,\hat{\varepsilon}_p+\hat{m}\,)\big]\;;\;(\hat{\varepsilon}_p=\hat{\beta}\,\varepsilon_p=\hat{\beta}\,\eta_p\,\varepsilon_+\;;\;\varepsilon_+>0)\;; && \text{(B.6)}\\
&= \hat{\beta}\hat{\gamma}^\mu\,\hat{\partial}_{p,\mu}+\imath\,\hat{\beta}\hat{\gamma}^\mu\,\hat{t}_\alpha\;\mathcal{V}^\mu_\alpha(x_p)+\hat{\beta}\,\hat{m}-\imath\,\varepsilon_p\;\hat{1}_{N_0\times N_0}\;; &&\\
\hat{H}^T(x_p) &= \big[\hat{\beta}(\hat{\partial\!\!\!/}_p+\imath\,\hat{\mathcal{V}\!\!\!/}(x_p)-\imath\,\hat{\varepsilon}_p+\hat{m}\,)\big]^T && \text{(B.7)}\\
&= -(\,\hat{\beta}\hat{\gamma}^\mu\,)^T\,\hat{\partial}_{p,\mu}-\imath\,(\,\hat{\beta}\hat{\gamma}^\mu\,)^T\,(-\hat{t}_\alpha^T)\;\mathcal{V}^\mu_\alpha(x_p)-(\,\hat{\beta}\,(-\hat{m})\,)^T-\imath\,\varepsilon_p\;\hat{1}_{N_0\times N_0}\;; &&\\
\hat{\mathfrak{B}}\;\hat{\Gamma}^\mu &= \begin{pmatrix}\hat{\beta}\;\hat{\gamma}^\mu & \\ & (\,\hat{\beta}\;\hat{\gamma}^\mu\,)^T\end{pmatrix}^{ba}\;;\;\hat{\mathfrak{B}}=\begin{pmatrix}\hat{\beta} & \\ & \hat{\beta}^T\end{pmatrix}^{ba}\;;\;\hat{\mathfrak{B}}\;\hat{M}=\begin{pmatrix}(\hat{\beta}\,\hat{m}) & \\ & (\hat{\beta}\,(-\hat{m})\,)^T\end{pmatrix}^{ba}\;; && \text{(B.8)}\\
\hat{\mathcal{V}\!\!\!/}(x_p) &= \hat{\Gamma}^\mu\;\hat{\mathcal{V}}_\mu(x_p)=\hat{\Gamma}^\mu\;\hat{\mathfrak{T}}_\alpha\;\mathcal{V}_{\alpha;\mu}(x_p)\;;\;\;\hat{\Gamma}^0=\begin{pmatrix}\hat{\gamma}^0 & 0\\ 0 & \hat{\gamma}^{0,T}\end{pmatrix}\;;\;\;\hat{\vec{\Gamma}}=\begin{pmatrix}\hat{\vec{\gamma}} & 0\\ 0 & -\hat{\vec{\gamma}}^T\end{pmatrix}\;; && \text{(B.9)}\\
\hat{\mathfrak{T}}_\alpha &= \begin{pmatrix}\hat{t}_\alpha & \\ & -\hat{t}_\alpha^T\end{pmatrix}^{ba}\;;\hat{\mathcal{V}}^\mu(x_p)=\hat{\mathfrak{T}}_\alpha\;\mathcal{V}^\mu_\alpha(x_p)\;; &&\\
\hat{\mathfrak{H}}(x_p) &= \hat{S}\Big(\hat{\mathfrak{B}}\;\hat{\Gamma}^\mu\;\hat{\partial}_{p,\mu}+\hat{\mathfrak{B}}\;\big(\imath\,\hat{\mathcal{V}\!\!\!/}(x_p)+\hat{M}\big)\Big)-\imath\,\varepsilon_p\;\hat{1}_{2N_0\times 2N_0}\;. && \text{(B.10)}
\end{aligned}
$$

If we consider the background potential $\mathcal{V}^\mu_\beta(x_p)$ (B.1) as a classical field, one can always choose a gauge condition, as Lorentz- (or axial-) gauge $(\,\hat{\partial}_{p,\mu}\mathcal{V}^\mu_\beta(x_p)\,)=0$, $(n_\nu\;\mathcal{V}^\nu_\beta(x_p)=0)$, respectively. If we assume that the chosen gauge condition of the composed field $\mathcal{V}^\mu_\beta(x_p)$ is violated, we can shift or adapt the auxiliary field $s_\alpha(x_p)$ for the axial gauge fixing of the 'original' gauge fields $A^\mu_\alpha(x_p)$ in such a manner that the chosen gauge condition is again attained (compare (B.1) with $s_\gamma(x_p)$). Therefore, one can also determine a gauge condition for $\mathcal{V}^\mu_\alpha(x_p)$ in advance, as far as classical fields are concerned.

In the quantum mechanical case, we obtain a Ward identity of the derived path integral (3.116) with background field averaging (3.59) and with projection $\hat{S}$ onto BCS-terms in the actions $\mathcal{A}_{DET}[\hat{T},\hat{\mathcal{V}\!\!\!/};\hat{\partial\!\!\!/}]$, $\mathcal{A}_{J_\psi}[\hat{T},\hat{\mathcal{V}\!\!\!/};\hat{\partial\!\!\!/}]$. Although there appears no action of a field strength tensor as $\hat{F}^{\mu\nu}_\alpha(x_p)$ (2.5) and Grassmann-valued fields $\psi_M(x_p)$ of quarks for the composed gauge field $\mathcal{V}^\mu_\alpha(x_p)$, as in the original case with field $A^\mu_\alpha(x_p)$, a gauge invariance follows because the change of actions with $\mathcal{V}^\mu_\alpha(x_p)$ in a gauge transformation is compensated by the change of the coset matrices $\hat{T}(x_p)=\exp\{-\hat{Y}(x_p)\,\}$. In this respect the actions $\mathcal{A}_{DET}[\hat{T},\hat{\mathcal{V}\!\!\!/};\hat{\partial\!\!\!/}]$, $\mathcal{A}_{J_\psi}[\hat{T},\hat{\mathcal{V}\!\!\!/};\hat{\partial\!\!\!/}]$ of the coset matrices replace the actions of a quadratic field strength tensor and of the anti-commuting Fermi fields for the composed gauge field $\mathcal{V}^\mu_\alpha(x_p)$. According to the definitions of Ref. [20] and of section 2, the gauge transformation (B.11) or its infinitesimal version (B.12,B.13) takes the form of the listed Eqs. (B.11-B.17) for the various BCS-fields and one-particle operators with $\mathrm{SU}_c(N_c=3)$ Lie group parameters $v_\beta(x_p)$, $\delta v_\beta(x_p)$

$$
\begin{aligned}
\psi_M(x_p) &\;\rightarrow\; \psi'_M(x_p)=\exp\{\imath\;{}_\alpha(x_p)\;\hat{t}_\alpha\}\;\psi_M(x_p)\;;\;\;\hat{t}_\alpha^\dagger=\hat{t}_\alpha\;; && \text{(B.11)}\\
\psi^*_M(x_p) &\;\rightarrow\; \psi^{*\prime}_M(x_p)=\exp\{-\imath\;{}_\alpha(x_p)\;\hat{t}_\alpha^T\}\;\psi^*_M(x_p)\;; &&\\
\psi_M(x_p) &\;\rightarrow\; \psi'_M(x_p)=\psi_M(x_p)+\delta\psi_M(x_p)\;; && \text{(B.12)}\\
\psi^*_M(x_p) &\;\rightarrow\; \psi^{*\prime}_M(x_p)=\psi_M(x_p)+\delta\psi^*_M(x_p)\;; &&\\
\delta\psi_M(x_p) &= \delta\psi_{f,m,r}(x_p)=\imath\;\delta v_\beta(x_p)\;\hat{t}_{\beta;rs}\;\psi_{f,m,s}(x_p)\;; && \text{(B.13)}\\
\delta\psi^*_M(x_p) &= \delta\psi^*_{f,m,r}(x_p)=\imath\;\delta v_\beta(x_p)\;(-\hat{t}^T_{\beta;rs})\;\psi^*_{f,m,s}(x_p)\;. &&
\end{aligned}
$$

Corresponding to the infinitesimal $\mathrm{SU}_c(N_c=3)$ Lie group transformations of quark fields, we list the infinitesimal changes of the composed gauge field $\hat{\mathcal{V}\!\!\!/}(x_p)$, of the anomalous doubled one-particle operator $\hat{\mathfrak{H}}(x_p)$, of the coset matrices $\hat{T}(x_p)$ of BCS quark pairs and of the source field $J^a_{\psi;M}(x_p)$

$$
\hat{\mathcal{V}\!\!\!/}(x_p) \;=\; \hat{\gamma}^\mu\;\hat{t}_\alpha\;\mathcal{V}_{\alpha;\mu}(x_p)\rightarrow\hat{\mathcal{V}\!\!\!/}(x_p)+\delta\hat{\mathcal{V}\!\!\!/}(x_p)\;; \qquad \text{(B.14)}
$$

$$\begin{aligned}
\delta\hat{\mathscr{V}}(x_p) &= \imath\,\delta v_\beta(x_p)\,[\hat{t}_\beta\,,\hat{\mathscr{V}}(x_p)\,]_- - \imath\,\big(\hat{\partial}_p\,\delta v_\beta(x_p)\,\big)\,\hat{t}_\beta\;; \\
\hat{\mathscr{V}}(x_p) &= \hat{\Gamma}^\mu\,\hat{\mathfrak{T}}_\alpha\,\mathcal{V}_{\alpha;\mu}(x_p) \to \hat{\mathscr{V}}(x_p) + \delta\hat{\mathscr{V}}(x_p)\;; \\
\delta\hat{\mathscr{V}}(x_p) &= \imath\,\delta v_\beta(x_p)\,[\hat{\mathfrak{T}}_\beta\,,\hat{\mathscr{V}}(x_p)\,]_- - \imath\,\hat{\Gamma}^\mu\big(\hat{\partial}_{p,\mu}\,\delta v_\beta(x_p)\,\big)\,\hat{\mathfrak{T}}_\beta\;; \\
\hat{\mathcal{H}}(x_p) &\to \hat{\mathcal{H}}(x_p) + \delta\hat{\mathcal{H}}(x_p)\;; \quad\to\ \delta\hat{\mathcal{H}}(x_p) = \imath\,\hat{\mathcal{B}}\,\hat{S}\,\delta\hat{\mathscr{V}}(x_p)\;; \\
\hat{T}^{ab}(x_p) &\to \hat{T}^{ab}(x_p) + \delta\hat{T}^{ab}(x_p)\;; \quad\to\ \delta\hat{T}^{ab}(x_p) = \imath\,\delta v_\beta(x_p)\,\big(\,[\hat{\mathfrak{T}}_\beta\,,\hat{T}(x_p)\,]_-\big)^{ab}\;; && \text{(B.15)} \\
\hat{T}^{-1;ab}(x_p) &\to \hat{T}^{-1;ab}(x_p) - \Big(\hat{T}^{-1}(x_p)\,\delta\hat{T}(x_p)\,\hat{T}^{-1}(x_p)\Big)^{ab}\;; \\
J_\psi^a(x_p) &\to J_\psi^a(x_p) + \delta J_\psi^a(x_p)\;; \quad\to\ \delta J_\psi^a(x_p) = \imath\,\delta v_\beta(x_p)\,\big(\hat{\mathfrak{T}}_\beta^{ab'}\;J_\psi^{b'}(x_p)\,\big)^a\;; && \text{(B.16)} \\
J_\psi^{\dagger,b}(x_p) &\to J_\psi^{\dagger,b}(x_p) + \delta J_\psi^{\dagger,b}(x_p)\;; \quad\to\ \delta J_\psi^{\dagger,b}(x_p) = -\imath\,\delta v_\beta(x_p)\,\big(J_\psi^{\dagger,a'}(x_p)\,\hat{\mathfrak{T}}_\beta^{a'b}\,\big)^b\;; \\
\hat{\mathcal{O}} &= \hat{T}^{-1}\,\hat{\mathcal{H}}\,\hat{T}\;; && \text{(B.17)} \\
\delta\hat{\mathcal{O}}(x_p) &= \delta\hat{T}^{-1}(x_p)\,\hat{\mathcal{H}}(x_p)\,\hat{T}(x_p) + \hat{T}^{-1}(x_p)\,\hat{\mathcal{H}}(x_p)\,\delta\hat{T}(x_p) + \hat{T}^{-1}(x_p)\,\delta\hat{\mathcal{H}}(x_p)\,\hat{T}(x_p) \\
&= \Big[\hat{T}^{-1}(x_p)\,\hat{\mathcal{H}}(x_p)\,\hat{T}(x_p)\,,\hat{T}^{-1}(x_p)\,\delta\hat{T}(x_p)\Big] + \hat{T}^{-1}(x_p)\,\imath\,\hat{\mathcal{B}}\,\hat{S}\,\delta\hat{\mathscr{V}}(x_p)\,\hat{T}(x_p)\;; \\
\delta\hat{\mathcal{O}}(x_p) &= \imath\,\delta v_\beta(x_p)\,\Big[\hat{\mathcal{O}}(x_p)\,,\hat{T}^{-1}(x_p)\,\big[\hat{\mathfrak{T}}_\beta\,,\hat{T}(x_p)\,\big]_-\Big]_- + \\
&+\ \imath\,\delta v_\beta(x_p)\,\hat{T}^{-1}(x_p)\,\imath\,\hat{\mathcal{B}}\,\hat{S}\,\big[\hat{\mathfrak{T}}_\beta\,,\hat{\mathscr{V}}(x_p)\,\big]_-\,\hat{T}(x_p) - \imath\,\big(\hat{\partial}_{p,\mu}\delta v_\beta(x_p)\big)\,\hat{T}^{-1}(x_p)\,\imath\,\hat{\mathcal{B}}\,\hat{S}\,\hat{\Gamma}^\mu\,\hat{\mathfrak{T}}_\beta\,\hat{T}(x_p)\,.
\end{aligned}$$

The variation of the effective generating functional (3.116) finally yields the conserved Ward identity (B.18) by expanding up to first order in $\delta v_\beta(x_p)$ of the gauge parameters where a partial integration has to be performed for the part $\delta\hat{\mathscr{V}}(x_p)$ (B.14) with derivatives $(\hat{\partial}_{p,\mu}\delta v_\beta(x_p)\,)$

$$\begin{aligned}
&\int_C d^4x_p\;\;\frac{\delta Z[\hat{\mathcal{J}},J_\psi,\hat{J}_{\psi\psi},\hat{\mathfrak{j}}^{(\hat{F})};(3.116)]}{\delta v_\beta(x_p)}\;\eta_p\;\delta v_\beta(x_p) = && \text{(B.18)} \\
&=\ \Bigg\langle\int_C d^4x_p\;\;\eta_p\;\delta v_\beta(x_p)\;\frac{\delta}{\delta v_\beta(x_p)}\Bigg(Z\Big[\hat{\mathscr{V}}(x_p);\hat{\mathfrak{S}}^{(\hat{F})},s_\alpha,\hat{\mathfrak{B}}_{\hat{F}},\hat{\mathfrak{b}}^{(\hat{F})},\hat{\mathcal{U}}_{\hat{F}},\hat{\mathfrak{v}}_{\hat{F}};\sigma_D;\hat{\mathfrak{j}}^{(\hat{F})};\textbf{Eq. (3.59)}\Big]\;\times \\
&\times\ \int d[\hat{T}^{-1}(x_p)\;d\hat{T}(x_p)]\;\;Z_{\hat{J}_{\psi\psi}}[\hat{T}]\;\exp\Big\{\mathcal{A}_{DET}[\hat{T},\hat{\mathscr{V}};\hat{\mathcal{J}}]\Big\}\;\exp\Big\{\imath\,\mathcal{A}_{J_\psi}[\hat{T},\hat{\mathscr{V}};\hat{\mathcal{J}}]\Big\}\Bigg)\Bigg\rangle = \\
&=\ \Big\langle Z\Big[\hat{\mathscr{V}}(x_p);\hat{\mathfrak{S}}^{(\hat{F})},s_\alpha,\hat{\mathfrak{B}}_{\hat{F}},\hat{\mathfrak{b}}^{(\hat{F})},\hat{\mathcal{U}}_{\hat{F}},\hat{\mathfrak{v}}_{\hat{F}};\sigma_D;\hat{\mathfrak{j}}^{(\hat{F})};\textbf{Eq. (3.59)}\Big]\;\times\;\int d[\hat{T}^{-1}(x_p)\;d\hat{T}(x_p)]\;\;Z_{\hat{J}_{\psi\psi}}[\hat{T}]\;\times \\
&\times\ \exp\Big\{\mathcal{A}_{DET}[\hat{T},\hat{\mathscr{V}};\hat{\mathcal{J}}] + \imath\,\mathcal{A}_{J_\psi}[\hat{T},\hat{\mathscr{V}};\hat{\mathcal{J}}]\Big\}\times\;\underset{\int_C d^4x_p\;\eta_p}{\overset{a(=1,2)}{\mathrm{TR}}}\;\;\underset{N_f,\hat{\gamma}_{mn}^{(\mu)},N_c}{\mathfrak{tr}}\Big\{\imath\,\delta v_\beta(x_p)\,\eta_p\;\times \\
&\times\ \Bigg[\Bigg(\Big[\hat{\mathcal{O}}(x_p)\,,\hat{T}^{-1}(x_p)\;\big[\hat{\mathfrak{T}}_\beta\,,\hat{T}(x_p)\,\big]_-\Big]_- + \hat{T}^{-1}(x_p)\,\imath\,\hat{\mathcal{B}}\,\hat{S}\,\big[\hat{\mathfrak{T}}_\beta\,,\hat{\mathscr{V}}(x_p)\,\big]_-\,\hat{T}(x_p)\Bigg)\times \\
&\times\ \Bigg(\langle\widehat{x_p}|\hat{\mathcal{O}}^{-1}|\widehat{x_p}\rangle + \frac{\imath}{2}\int_C d^4y_q\;d^4y'_{q'}\;\langle\widehat{x_p}|\hat{\mathcal{O}}^{-1}|\widehat{y_q}\rangle\;\hat{T}^{-1}(y_q)\;\hat{I}\;J_\psi(y_q)\;\otimes\;J_\psi^\dagger(y'_{q'})\;\hat{I}\;\hat{T}(y'_{q'})\;\langle\widehat{y'_{q'}}|\hat{\mathcal{O}}^{-1}|\widehat{x_p}\rangle\Bigg) + \\
&+\ \Bigg(\hat{\mathscr{\partial}}_{p,\mu}\;\hat{T}^{-1}(x_p)\,\imath\,\hat{\mathcal{B}}\,\hat{S}\,\hat{\Gamma}^\mu\,\hat{\mathfrak{T}}_\beta\,\hat{T}(x_p)\times \\
&\times\ \Bigg(\langle\widehat{x_p}|\hat{\mathcal{O}}^{-1}|\widehat{x_p}\rangle + \frac{\imath}{2}\int_C d^4y_q\;d^4y'_{q'}\;\langle\widehat{x_p}|\hat{\mathcal{O}}^{-1}|\widehat{y_q}\rangle\;\hat{T}^{-1}(y_q)\;\hat{I}\;J_\psi(y_q)\;\otimes\;J_\psi^\dagger(y'_{q'})\;\hat{I}\;\hat{T}(y'_{q'})\;\langle\widehat{y'_{q'}}|\hat{\mathcal{O}}^{-1}|\widehat{x_p}\rangle\Bigg)\Bigg) + \\
&+\ \frac{\imath}{2}\int_C d^4y_q\Bigg(\langle\widehat{x_p}|\hat{\mathcal{O}}^{-1}|\widehat{y_q}\rangle\;\hat{T}^{-1}(y_q)\;\hat{I}\;J_\psi(y_q)\otimes J_\psi^\dagger(x_p)\;\hat{I}\;\hat{T}(x_p)\;\hat{\mathfrak{T}}_\beta +
\end{aligned}$$

$$- \quad \hat{\mathfrak{I}}_\beta\,\hat{T}^{-1}(x_p)\,\hat{I}\,J_\psi(x_p)\otimes J_\psi^\dagger(y_q)\,\hat{I}\,\hat{T}(y_q)\,\langle\widehat{y_q}|\hat{\mathfrak{O}}^{-1}|\widehat{x_p}\rangle\Big)\Big]\Big]\Big\}\Big\rangle\,.$$

C Derivation of the chiral anomaly from the change of integration variables

C.1 Axial '$\hat{\gamma}_5$' transformations of the actions in the exponentials

In the following we describe the derivation of the chiral anomaly according to [33, 34]. It is finally obtained from the change of the fermionic path integration variables $d[\psi^\dagger(x_p)]\;d[\psi(x_p)]$ where the divergence of the corresponding Jacobian is determined by a gauge invariant cut-off regulator and has therefore to be regarded as an additional quantum phenomenon, violating the classical, (massless), conserved '$\hat{\gamma}_5$' Noether current relation. These chiral anomalies are also related to nontrivial differential topologies which allow for explicit derivations in a geometrical context [35]. The original derivation of the anomaly is given within perturbation series from the regularization of a triangle diagram with the axial '$\hat{\gamma}_5$' matrix [34]. In this subsection C.1 we briefly restrict to the change of the original QCD action $\mathcal{A}[\psi,\hat{D}_\mu\psi,\hat{F}]$ and the action $\mathcal{A}_S[\hat{\mathcal{J}},J_\psi,\hat{J}_{\psi\psi},\hat{\jmath}^{(\hat{F})}]$ of source terms under axial $U_A(1)$ transformations $\hat{g}_A(x_p)$, $\hat{g}_A^\dagger(x_p)$ (C.5-C.7) of the Grassmann fields $\psi_{f,m,r}(x_p)$ (C.1-C.4) with '$\hat{\gamma}_5$' Dirac matrix and isospin-(flavour-) generator $\hat{\mathfrak{t}}_{0,fg}$ and angle $\alpha(x_p)$. The finite, axial $U_A(1)$ transformations are listed in relations (C.1-C.7) with their actions onto the fermionic fields

$$\begin{aligned}
\psi_{f,m,r}(x_p) &= \delta_{rs}\Big[\exp\Big\{\imath\,\alpha(x_p)\;\hat{\mathfrak{t}}_{0,f'g'}\;\big(\hat{\gamma}_5\big)_{m'n'}\;\delta_{r's'}\Big\}\Big]_{f,m,r;g,n,s}\;\psi'_{g,n,s}(x_p)\;; &\text{(C.1)}\\
\psi(x_p) &= \hat{g}_A(x_p)\;\psi'(x_p)\;; &\text{(C.2)}\\
\psi^\dagger(x_p) &= \psi'^\dagger(x_p)\;\hat{g}_A^\dagger(x_p)\;; &\text{(C.3)}\\
\bar{\psi}(x_p) &= \psi^\dagger(x_p)\;\hat{\beta}=\psi'^\dagger(x_p)\;\hat{g}_A^\dagger(x_p)\;\hat{\beta}=\bar{\psi}'(x_p)\;\hat{g}_A(x_p)\;; &\text{(C.4)}\\
\hat{g}_A(x_p) &= \cos\Big(\alpha(x_p)\;\hat{\mathfrak{t}}_0\Big)+\imath\,\hat{\gamma}_5\;\sin\Big(\alpha(x_p)\;\hat{\mathfrak{t}}_0\Big)\;;\quad \hat{\mathfrak{t}}_0^\dagger=\hat{\mathfrak{t}}_0\;; &\text{(C.5)}\\
\hat{g}_A^\dagger(x_p) &= \cos\Big(\alpha(x_p)\;\hat{\mathfrak{t}}_0\Big)-\imath\,\hat{\gamma}_5\;\sin\Big(\alpha(x_p)\;\hat{\mathfrak{t}}_0\Big)\;;\quad \hat{\mathfrak{t}}_{0;f,m,r;g,n,s}=\hat{\mathfrak{t}}_{0,fg}\;\delta_{mn}\;\delta_{rs}\;; &\text{(C.6)}\\
\hat{\gamma}_5^\dagger &= \hat{\gamma}_5\;;\quad \hat{\gamma}_5^T=\hat{\gamma}_5\;;\quad \big(\hat{\gamma}_5\big)^2=\hat{1}_{4\times4}\;;\quad \big\{\hat{\beta}\;,\hat{\gamma}_5\big\}_+=0\;;\quad \big\{\hat{\gamma}^\mu\;,\hat{\gamma}_5\big\}_+=0\,. &\text{(C.7)}
\end{aligned}$$

The anti-commuting property (C.7) of the axial '$\hat{\gamma}_5$' matrix with the Dirac matrices $\hat{\gamma}^\mu$ has to be emphasized because this particular property causes to transform the fermi field $\bar{\psi}(x_p)$ (C.4) with the same matrix $\hat{g}_A(x_p)$ as for $\psi(x_p)$ (C.2) instead of $\hat{g}_A^\dagger(x_p)$ as for $\psi^\dagger(x_p)$ (C.3). We insert the transformed fields $\psi'_{g,n,s}(x_p)$ (C.1-C.4) into the original actions $\mathcal{A}[\psi,\hat{D}_\mu\psi,\hat{F}]$ and $\mathcal{A}_S[\hat{\mathcal{J}},J_\psi,\hat{J}_{\psi\psi},\hat{\jmath}^{(\hat{F})}]$ for a finite angle $\alpha(x_p)$ with the transformation matrices $\hat{g}_A(x_p)$, $\hat{g}_A^\dagger(x_p)$ (C.5-C.7) and expand to first order $\delta\alpha(x_p)$ for infinitesimal values, using the $U_A(1)$ Lie group properties. This results into $\delta\mathcal{A}[\psi',\hat{D}_\mu\psi',\hat{F}]$ (C.8-C.10) and $\delta\mathcal{A}'_S[\hat{\mathcal{J}},J_\psi,\hat{J}_{\psi\psi},\hat{\jmath}^{(\hat{F})}]$ (C.11-C.13) where a partial integration has to be performed within $\delta\mathcal{A}[\psi',\hat{D}_\mu\psi',\hat{F}]$ in order to attain only the first order variation $\delta\alpha(x_p)$ without any derivative

$$\begin{aligned}
&\mathcal{A}[\psi,\hat{D}_\mu\psi,\hat{F}]=\int_C d^4x_p\Big\{-\frac{1}{4}\hat{F}_{\alpha;\mu\nu}(x_p)\;\hat{F}_\alpha^{\mu\nu}(x_p)+ &\text{(C.8)}\\
&-\quad\psi'^\dagger(x_p)\;\hat{g}_A^\dagger(x_p)\Big(\hat{\beta}\,\hat{\gamma}^\mu\;\hat{\partial}_{p,\mu}-\imath\,g\,\hat{\beta}\,\hat{\gamma}^\mu\;\hat{t}_\alpha\;A_{\alpha;\mu}(x_p)+\hat{\beta}\;\hat{m}-\imath\,\varepsilon_+\;\eta_p\Big)\hat{g}_A(x_p)\;\psi'(x_p)\Big\}=\\
&=\quad\int_C d^4x_p\Big\{-\frac{1}{4}\hat{F}_{\alpha;\mu\nu}(x_p)\;\hat{F}_\alpha^{\mu\nu}(x_p)-\bar{\psi}'(x_p)\;\hat{g}_A(x_p)\Big(\hat{\not{\partial}}_p-\imath\,g\,\hat{\not{A}}+\hat{m}-\imath\,\hat{\beta}\;\varepsilon_+\;\eta_p\Big)\hat{g}_A(x_p)\;\psi'(x_p)\Big\}\;;\\
&\mathcal{A}[\psi,\hat{D}_\mu\psi,\hat{F}]=\mathcal{A}[\psi',\hat{D}_\mu\psi',\hat{F}]+\delta\mathcal{A}[\psi',\hat{D}_\mu\psi',\hat{F}]\;; &\text{(C.9)}\\
&\delta\mathcal{A}[\psi',\hat{D}_\mu\psi',\hat{F}]=-\imath\int_C d^4x_p\Big\{\bar{\psi}'(x_p)\;\hat{\gamma}_5\big\{\hat{\mathfrak{t}}_0\;,\hat{m}\big\}_+\psi'(x_p)\;\delta\alpha(x_p)+\bar{\psi}'(x_p)\;\hat{\gamma}^\mu\Big(\hat{\partial}_{p,\mu}\delta\alpha(x_p)\Big)\hat{\gamma}_5\;\hat{\mathfrak{t}}_0\;\psi'(x_p)\Big\}= &\text{(C.10)}\\
&=\quad\imath\int_C d^4x_p\Big\{\hat{\partial}_{p,\mu}\Big(\bar{\psi}'(x_p)\;\hat{\gamma}^\mu\;\hat{\gamma}_5\;\hat{\mathfrak{t}}_0\;\psi'(x_p)\Big)-\bar{\psi}'(x_p)\;\hat{\gamma}_5\;\big\{\hat{\mathfrak{t}}_0\;,\hat{m}\big\}_+\;\psi'(x_p)\Big\}\;\delta\alpha(x_p)\;;
\end{aligned}$$

$$\mathcal{A}_S[\hat{\jmath}, J_\psi, \hat{J}_{\psi\psi}, \hat{\mathrm{j}}^{(\hat{F})}] = \int_C d^4x_p \bigg\{ \Big(j_\psi^\dagger(x_p)\, \hat{g}_A(x_p)\, \psi'(x_p) + \psi'^\dagger(x_p)\, \hat{g}_A^\dagger(x_p)\, j_\psi(x_p) \Big) + \tag{C.11}$$
$$+ \frac{1}{2}\bigg[\Big(\hat{g}_A(x_p)\, \psi'(x_p)\Big)^T \hat{j}_{\psi\psi}^\dagger(x_p) \Big(\hat{g}_A(x_p)\, \psi'(x_p)\Big) + \Big(\hat{g}_A(x_p)\, \psi'(x_p)\Big)^\dagger \hat{j}_{\psi\psi}(x_p) \Big(\hat{g}_A(x_p)\, \psi'(x_p)\Big)^* \bigg] +$$
$$+ \frac{1}{2}\int_C d^4y_q\, \Psi'^\dagger(y_q) \begin{pmatrix} \hat{g}_A^\dagger(y_q) & \\ & \hat{g}_A^T(y_q) \end{pmatrix} \hat{\jmath}(y_q, x_p) \begin{pmatrix} \hat{g}_A(x_p) & \\ & \hat{g}_A^*(x_p) \end{pmatrix} \Psi'(x_p) + \hat{j}_{\hat{F};\alpha;\mu\nu}(x_p)\, \hat{F}_\alpha^{\mu\nu}(x_p) \bigg\} ;$$

$$\mathcal{A}_S[\hat{\jmath}, J_\psi, \hat{J}_{\psi\psi}, \hat{\mathrm{j}}^{(\hat{F})}] = \mathcal{A}'_S[\hat{\jmath}, J_\psi, \hat{J}_{\psi\psi}, \hat{\mathrm{j}}^{(\hat{F})}] + \delta\mathcal{A}'_S[\hat{\jmath}, J_\psi, \hat{J}_{\psi\psi}, \hat{\mathrm{j}}^{(\hat{F})}] ; \tag{C.12}$$

$$\delta\mathcal{A}'_S[\hat{\jmath}, J_\psi, \hat{J}_{\psi\psi}, \hat{\mathrm{j}}^{(\hat{F})}] = \imath \int_C d^4x_p\;\; \delta\alpha(x_p) \bigg\{ \Big(j_\psi^\dagger(x_p)\, \hat{\gamma}_5\, \hat{\mathfrak{t}}_0\, \psi'(x_p) - \psi'^\dagger(x_p)\, \hat{\gamma}_5\, \hat{\mathfrak{t}}_0\, j_\psi(x_p) \Big) + \tag{C.13}$$
$$+ \frac{1}{2}\bigg[\psi'^T(x_p) \Big(\hat{j}_{\psi\psi}^\dagger(x_p)\, \hat{\gamma}_5\, \hat{\mathfrak{t}}_0 + \hat{\gamma}_5\, \hat{\mathfrak{t}}_0^T\, \hat{j}_{\psi\psi}^\dagger(x_p) \Big) \psi'(x_p) - \psi'^\dagger(x_p) \Big(\hat{j}_{\psi\psi}(x_p)\, \hat{\gamma}_5\, \hat{\mathfrak{t}}_0^* + \hat{\gamma}_5\, \hat{\mathfrak{t}}_0\, \hat{j}_{\psi\psi}(x_p) \Big) \psi'^*(x_p) \bigg] +$$
$$+ \frac{1}{2}\int_C d^4y_q \bigg[\Psi'^\dagger(y_q)\, \hat{\jmath}(y_q, x_p) \begin{pmatrix} \hat{\gamma}_5\, \hat{\mathfrak{t}}_0 & \\ & -\hat{\gamma}_5\, \hat{\mathfrak{t}}_0^* \end{pmatrix} \Psi'(x_p) + \Psi'^\dagger(x_p) \begin{pmatrix} -\hat{\gamma}_5\, \hat{\mathfrak{t}}_0 & \\ & \hat{\gamma}_5\, \hat{\mathfrak{t}}_0^T \end{pmatrix} \hat{\jmath}(x_p, y_q)\, \Psi'(x_p) \bigg] \bigg\} .$$

After reordering the first order variations with $\delta\alpha(x_p)$, we achieve the axial current relation (C.14) of $\bar{\psi}(x_p)\, \hat{\gamma}^\mu\, \hat{\gamma}_5\, \hat{\mathfrak{t}}_0\, \psi(x_p)$ whose conservation is perturbed by the mass term $\bar{\psi}(x_p)\, \hat{\gamma}_5 \left\{ \hat{\mathfrak{t}}_0 \,\boldsymbol{,}\, \hat{m} \right\}_+ \psi(x_p)$ and the source parts which are also determined by the isospin-(flavour-) generator $\hat{\mathfrak{t}}_{0,fg}$

$$\delta\mathcal{A}[\psi', \hat{D}_\mu\psi', \hat{F}] - \delta\mathcal{A}'_S[\hat{\jmath}, J_\psi, \hat{J}_{\psi\psi}, \hat{\mathrm{j}}^{(\hat{F})}] \equiv 0 \,; \Longrightarrow \tag{C.14}$$
$$\hat{\partial}_{p,\mu} \Big(\bar{\psi}(x_p)\, \hat{\gamma}^\mu\, \hat{\gamma}_5\, \hat{\mathfrak{t}}_0\, \psi(x_p) \Big) = \bar{\psi}(x_p)\, \hat{\gamma}_5 \left\{ \hat{\mathfrak{t}}_0 \,\boldsymbol{,}\, \hat{m} \right\}_+ \psi(x_p) + j_\psi^\dagger(x_p)\, \hat{\gamma}_5\, \hat{\mathfrak{t}}_0\, \psi(x_p) - \psi^\dagger(x_p)\, \hat{\gamma}_5\, \hat{\mathfrak{t}}_0\, j_\psi(x_p) +$$
$$+ \frac{1}{2}\bigg[\psi^T(x_p) \Big(\hat{j}_{\psi\psi}^\dagger(x_p)\, \hat{\gamma}_5\, \hat{\mathfrak{t}}_0 + \hat{\gamma}_5\, \hat{\mathfrak{t}}_0^T\, \hat{j}_{\psi\psi}^\dagger(x_p) \Big) \psi(x_p) - \psi^\dagger(x_p) \Big(\hat{j}_{\psi\psi}(x_p)\, \hat{\gamma}_5\, \hat{\mathfrak{t}}_0^* + \hat{\gamma}_5\, \hat{\mathfrak{t}}_0\, \hat{j}_{\psi\psi}(x_p) \Big) \psi^*(x_p) \bigg] +$$
$$+ \frac{1}{2}\int_C d^4y_q \bigg[\Psi^\dagger(y_q, x_p)\, \hat{\jmath}(y_q, x_p) \begin{pmatrix} \hat{\gamma}_5\, \hat{\mathfrak{t}}_0 & \\ & -\hat{\gamma}_5\, \hat{\mathfrak{t}}_0^* \end{pmatrix} \Psi(x_p) + \Psi^\dagger(x_p) \begin{pmatrix} -\hat{\gamma}_5\, \hat{\mathfrak{t}}_0 & \\ & \hat{\gamma}_5\, \hat{\mathfrak{t}}_0^T \end{pmatrix} \hat{\jmath}(x_p, y_q)\, \Psi(y_q) \bigg] .$$

C.2 Calculation of the Jacobian with a gauge invariant cut-off regulator

Apart from the transformation of the actions $\mathcal{A}[\psi, \hat{D}_\mu\psi, \hat{F}]$, $\mathcal{A}_S[\hat{\jmath}, J_\psi, \hat{J}_{\psi\psi}, \hat{\mathrm{j}}^{(\hat{F})}]$ as phases in the exponential, one has also to consider the change of the Jacobian under transformations (C.15,C.16) with matrix '$\hat{\gamma}_5$' and the parameter $\alpha(x_p)$ for the generator $\hat{\mathfrak{t}}_{0,fg}$ of axial isospin-(flavour-) rotations

$$\psi'(x_p) \;=\; \exp\big\{ -\imath\, \alpha(x_p)\, \hat{\gamma}_5\, \hat{\mathfrak{t}}_0 \big\}\; \psi(x_p) \,; \tag{C.15}$$
$$\bar{\psi}'(x_p) \;=\; \bar{\psi}(x_p)\; \exp\big\{ -\imath\, \alpha(x_p)\, \hat{\gamma}_5\, \hat{\mathfrak{t}}_0 \big\} \,. \tag{C.16}$$

We expand the fermionic fields $\psi(x_p)$, $\bar{\psi}(x_p)$ in terms of orthonormalized eigenfunctions $\varphi_L(x_p)$ of the (massless) gauge invariant derivative $\hat{\not{D}}_p$ with eigenvalues $-\imath\, e_L^{(p)}$ on the time contour '$p=\pm$' and have to include Dirac spinors $\chi_L^{(p)}$, $\bar{\chi}_L^{(p)}$ in order to regard the anti-commuting character of the fermi fields $\psi(x_p)$, $\bar{\psi}(x_p)$ (C.17-C.19). One might infer that the eigenfunctions $\varphi_L(x_p)$ cannot be taken as ordinary, complex-valued, scalar functions, due to the matrix property of $\hat{\not{D}}_p$, but an appropriate unitary transformation '$\hat{U}$' can always be chosen to rotate to scalar, complex-valued eigenfunctions $\varphi_L(x_p)$, leaving the Dirac spinor property of $\psi(x_p)$, $\bar{\psi}(x_p)$ entirely within $\chi_L^{(p)}$, $\bar{\chi}_L^{(p)}$ (C.20,C.21). Using the unitary invariance of the 'massless' derivative operator $\hat{\not{D}}_p$ (C.22,C.23), we can eventually compute the expectation value $\bar{\psi}(x_p)\,(\hat{\not{D}}_p + \hat{m}\,)\,\psi(x_p)$ (C.24) in terms of the Dirac spinors $\chi_L^{(p)}$, $\bar{\chi}_L^{(p)}$ and eigenvalues $-\imath\, e_L^{(p)}$

$$\psi(x_p) \;=\; \sum_L \varphi_L(x_p)\; \chi_L^{(p)} = \sum_L \langle x_p | L \rangle\; \chi_L^{(p)} \,; \tag{C.17}$$

$$\bar{\psi}(x_p) = \sum_L \bar{\chi}_L^{(p)} \, \varphi_L^\dagger(x_p) = \sum_L \bar{\chi}_L^{(p)} \, \langle L|x_p\rangle \; ; \tag{C.18}$$

$$\chi_L^{(p)} \, , \, \bar{\chi}_L^{(p)} := \text{Dirac spinors} \;\; ; \qquad \bar{\chi}_L^{(p)} = \chi_L^{(p)\dagger} \, \hat{\beta} \; ; \tag{C.19}$$

$$\hat{U} \hat{\not{D}}_p \, \hat{U}^\dagger \left(\hat{U}\varphi_L(x_p)\right) = \hat{U}\Big(\hat{\not{\partial}}_p - \imath \, g \hat{\not{A}}(x_p)\Big)\hat{U}^\dagger \left(\hat{U}\varphi_L(x_p)\right) = -\imath \, e_L^{(p)} \, \left(\hat{U}\varphi_L(x_p)\right) \; ; \tag{C.20}$$

$$\text{unitary transformation } \left(\hat{U}\varphi_L(x_p)\right) \;\; \to \;\; \text{for scalar, complex-valued eigenfunctions } \varphi_L(x_p) \; ; \tag{C.21}$$

$$\hat{\not{D}}_p \varphi_L(x_p) = \Big(\hat{\not{\partial}}_p - \imath \, g \hat{\not{A}}(x_p)\Big)\varphi_L(x_p) = -\imath \, e_L^{(p)} \, \varphi_L(x_p) \; ; \tag{C.22}$$

$$\delta_{K,L} = \int d^4x_p \, \varphi_K^\dagger(x_p) \, \varphi_L(x_p) \; ; \tag{C.23}$$

$$\int d^4x_p \, \bar{\psi}(x_p) \left(\hat{\not{D}}_p + \hat{m}\right) \psi(x_p) = \lim_{N_L\to\infty} \sum_{L=1}^{N_L} \bar{\chi}_L^{(p)} \, \left(-\imath \, e_L^{(p)} + \hat{m}\right) \chi_L^{(p)} \; . \tag{C.24}$$

The change with the Jacobian of $d[\psi^\dagger(x_p)] \; d[\psi(x_p)]$ to $d\bar{\chi}_L^{(p)} \; d\chi_L^{(p)}$ (C.25) follows from the inverse of the transformation matrices $\langle L|x_p\rangle$, $\langle x_p|L\rangle$, due to the anti-commuting property of the fermionic fields

$$\begin{aligned} d[\psi^\dagger(x_p)] \; d[\psi(x_p)] &\overset{\det(\hat{\beta})=1}{=} d[\bar{\psi}(x_p)] \; d[\psi(x_p)] = \prod_{p=\pm} \lim_{N_L\to\infty} \prod_{L=1}^{N_L} d\bar{\chi}_L^{(p)} \; d\chi_L^{(p)} \Big[\det\big(\langle L|x_p\rangle\big) \; \det\big(\langle x_p|L\rangle\big)\Big]^{-1} \\ &= \prod_{p=\pm} \lim_{N_L\to\infty} \prod_{L=1}^{N_L} d\bar{\chi}_L^{(p)} \; d\chi_L^{(p)} \; \Big[\det\Big(\int d^4x_p \langle L|x_p\rangle\langle x_p|K\rangle\Big)\Big]^{-1} \\ &= \prod_{p=\pm} \lim_{N_L\to\infty} \prod_{L=1}^{N_L} d\bar{\chi}_L^{(p)} \; d\chi_L^{(p)} \; \Big[\det\big(\delta_{L,K}\big)\Big]^{-1} = \prod_{p=\pm} \lim_{N_L\to\infty} \prod_{L=1}^{N_L} d\bar{\chi}_L^{(p)} \; d\chi_L^{(p)} \; . \end{aligned} \tag{C.25}$$

We apply the axial transformations (C.1-C.7,C.26,C.27) from $\psi(x_p)$, $\bar{\psi}(x_p)$, $(\chi_L^{(p)}, \, \bar{\chi}_L^{(p)})$ to $\psi'(x_p)$, $\bar{\psi}'(x_p)$ (C.28,C.30) and, respectively, to the transformed Dirac spinors $\chi_L'^{(p)}$, $\bar{\chi}_L'^{(p)}$ and obtain the corresponding first order variation $\delta\alpha(x_p)$ between $\chi_L'^{(p)}$, $\bar{\chi}_L'^{(p)}$ and $\chi_L^{(p)}$, $\bar{\chi}_L^{(p)}$ (C.29,C.31) which has to be substituted in the integration measure (C.32). This specifies the Jacobian for the change from $\chi_L'^{(p)}$, $\bar{\chi}_L'^{(p)}$ to $\chi_L^{(p)}$, $\bar{\chi}_L^{(p)}$, and according to relations (C.17-C.24) and especially (C.25), also the change (C.33) with the Jacobian of (C.32) for the transformation from $\psi'(x_p)$, $\bar{\psi}'(x_p)$ to $\psi(x_p)$, $\bar{\psi}(x_p)$

$$\psi'(x_p) = \psi(x_p) - \imath \, \hat{\mathfrak{t}}_0 \, \hat{\gamma}_5 \, \psi(x_p) \, \delta\alpha(x_p) \; ; \tag{C.26}$$

$$\bar{\psi}'(x_p) = \bar{\psi}(x_p) - \imath \, \bar{\psi}(x_p) \, \hat{\mathfrak{t}}_0 \, \hat{\gamma}_5 \, \delta\alpha(x_p) \; ; \tag{C.27}$$

$$\psi'(x_p) = \sum_L \varphi_L(x_p) \, \chi_L'^{(p)} = \sum_L \varphi_L(x_p) \, \chi_L^{(p)} - \imath \, \delta\alpha(x_p) \, \hat{\mathfrak{t}}_0 \, \hat{\gamma}_5 \sum_L \varphi_L(x_p) \, \chi_L^{(p)} \; ; \tag{C.28}$$

$$\chi_L'^{(p)} = \chi_L^{(p)} - \imath \sum_K \left(\int d^4x_p \, \varphi_L^\dagger(x_p) \, \hat{\mathfrak{t}}_0 \, \hat{\gamma}_5 \, \varphi_K(x_p) \, \delta\alpha(x_p)\right) \chi_K^{(p)} \; ; \tag{C.29}$$

$$\bar{\psi}'(x_p) = \sum_L \bar{\chi}_L'^{(p)} \, \varphi_L^\dagger(x_p) = \sum_L \bar{\chi}_L^{(p)} \, \varphi_L^\dagger(x_p) - \imath \, \delta\alpha(x_p) \sum_L \bar{\chi}_L^{(p)} \, \varphi_L^\dagger(x_p) \, \hat{\mathfrak{t}}_0 \, \hat{\gamma}_5 \; ; \tag{C.30}$$

$$\bar{\chi}_L'^{(p)} = \bar{\chi}_L^{(p)} - \imath \sum_K \bar{\chi}_K^{(p)} \left(\int d^4x_p \, \varphi_K^\dagger(x_p) \, \hat{\mathfrak{t}}_0 \, \hat{\gamma}_5 \, \varphi_L(x_p) \, \delta\alpha(x_p)\right) \; ; \tag{C.31}$$

$$\begin{aligned} \prod_{p=\pm} \prod_{L=1}^{N_L} d\bar{\chi}_L'^{(p)} \; d\chi_L'^{(p)} &= \prod_{p=\pm} \prod_{L=1}^{N_L} d\bar{\chi}_L^{(p)} \; d\chi_L^{(p)} \;\times \\ &\times \;\; \det\Big[\delta_{L,K} \, \overbrace{\delta_{f,m,r;g,n,s}}^{\delta_{M;N}} - \imath \Big(\int d^4x_p \, \delta\alpha(x_p) \, \varphi_L^\dagger(x_p) \, \hat{\mathfrak{t}}_{0,fg} \, (\hat{\gamma}_5)_{mn} \, \delta_{rs} \, \varphi_K(x_p)\Big)\Big]^{-2} \; ; \end{aligned} \tag{C.32}$$

$$d[\psi'^{\dagger}(x_p)]\;d[\psi'(x_p)] \overset{\det(\hat{\beta})}{=} d[\bar{\psi}'(x_p)]\;d[\psi'(x_p)] = \prod_{p=\pm}\lim_{N_L\to\infty} d\bar{\chi}_L'^{(p)}\;d\chi_L'^{(p)} = \tag{C.33}$$
$$= \prod_{p=\pm}\bigg(\lim_{N_L\to\infty}\prod_{L=1}^{N_L} d\bar{\chi}_L^{(p)}\;d\chi_L^{(p)}\bigg)\exp\bigg\{2\imath\lim_{N_L\to\infty}\sum_{L=1}^{N_L}\int d^4x_p\;\delta\alpha(x_p)\underset{N_f,\hat{\gamma}_{mn}^{(\mu)},N_c}{\mathfrak{tr}}\Big[\varphi_L^{\dagger}(x_p)\;\hat{\mathfrak{t}}_0\;\hat{\gamma}_5\;\varphi_L(x_p)\Big]\bigg\} =$$
$$= d[\bar{\psi}(x_p)]\;d[\psi(x_p)]\;\prod_{p=\pm}\mathfrak{J}_p\;.$$

We separate the Jacobian $\mathfrak{J}_p$ (on the time contour '$p=\pm$') (C.33,C.34) from the integration variables and have to examine a trace relation of the complex-valued, scalar eigenfunctions $\varphi_L(x_p)$ of the gauge invariant derivative $\hat{\not{D}}_p$

$$\mathfrak{J}_p = \exp\Big\{2\imath\,'\mathsf{exp}'\Big\} = \exp\bigg\{2\imath\lim_{N_L\to\infty}\sum_{L=1}^{N_L}\int d^4x_p\;\delta\alpha(x_p)\underset{N_f,\hat{\gamma}_{mn}^{(\mu)},N_c}{\mathfrak{tr}}\Big[\varphi_L^{\dagger}(x_p)\;\hat{\mathfrak{t}}_0\;\hat{\gamma}_5\;\varphi_L(x_p)\Big]\bigg\}\;. \tag{C.34}$$

Apparently, the phase ''exp'' within the Jacobian $\mathfrak{J}_p$ (C.34) diverges due to the trace operations of the eigenfunctions $\varphi_L(x_p)$; therefore, one transforms to the momentum representation $\widetilde{\varphi}_L(k_p)$, $\widetilde{\varphi}_L^{\dagger}(k_p')$ (C.36) of $\varphi_L(x_p)\,\varphi_L^{\dagger}(x_p)$ and introduces a gauge invariant cut-off regulator $\exp\{\hat{\not{D}}_p^2/M^2\}$ of the original operator $\hat{\not{D}}_p$ (C.17-C.24) for the eigenfunctions $\varphi_L(x_p)$, modified by the 'regulating' mass parameter 'M'

$$'\mathsf{exp}' = \lim_{N_L\to\infty}\sum_{L=1}^{N_L}\int d^4x_p\;\delta\alpha(x_p)\underset{N_f,\hat{\gamma}_{mn}^{(\mu)},N_c}{\mathfrak{tr}}\Big[\varphi_L^{\dagger}(x_p)\;\hat{\mathfrak{t}}_0\;\hat{\gamma}_5\;\varphi_L(x_p)\Big] = \tag{C.35}$$
$$= \lim_{M^2\to\infty}\sum_{L=1}^{\infty}\int d^4x_p\frac{d^4k_p}{(2\pi)^4}\frac{d^4k_p'}{(2\pi)^4}\;\delta\alpha(x_p)\underset{N_f,\hat{\gamma}_{mn}^{(\mu)},N_c}{\mathfrak{tr}}\Big[\widetilde{\varphi}_L^{\dagger}(k_p')\;e^{-\imath\,k_p'\cdot x_p}\;\hat{\mathfrak{t}}_0\;\hat{\gamma}_5\;\exp\{\hat{\not{D}}_p^2/M^2\}\;e^{\imath\,k_p\cdot x_p}\;\widetilde{\varphi}_L(k_p)\Big]\;;$$
$$\sum_{L=1}^{\infty}\widetilde{\varphi}_L(k_p)\;\widetilde{\varphi}_L^{\dagger}(k_p') = (2\pi)^4\;\delta^{(4)}(k_p-k_p')\;\;. \tag{C.36}$$

Using the completeness relation (C.36) for (C.35), we have exchanged the eigenfunctions $\varphi_L(x_p)$, $\varphi_L^{\dagger}(x_p)$ by plane waves $e^{\imath\,k_p\cdot x_p}$, $e^{-\imath\,k_p\cdot x_p}$ and obtain the trace in momentum representation where the divergence is reduced by the operator $\exp\{\hat{\not{D}}_p^2/M^2\}$ with mass parameter M

$$'\mathsf{exp}' = \lim_{M^2\to\infty}\int d^4x_p\frac{d^4k_p}{(2\pi)^4}\;\delta\alpha(x_p)\underset{N_f,\hat{\gamma}_{mn}^{(\mu)},N_c}{\mathfrak{tr}}\Big[e^{-\imath\,k_p\cdot x_p}\;\hat{\mathfrak{t}}_0\;\hat{\gamma}_5\;\exp\{\hat{\not{D}}_p^2/M^2\}\;e^{\imath\,k_p\cdot x_p}\Big]\;; \tag{C.37}$$
$$\hat{\not{D}}_p^2 = \hat{\gamma}^{\mu}\,\hat{\gamma}^{\nu}\,\hat{D}_{p,\mu}\,\hat{D}_{p,\nu} = \hat{D}_{p,\mu}\hat{D}_p^{\mu} + \frac{1}{2}\,\hat{\gamma}^{\mu}\,\hat{\gamma}^{\nu}\,\big[\hat{D}_{p,\mu}\;,\;\hat{D}_{p,\nu}\big]_- \tag{C.38}$$
$$= \hat{D}_{p,\mu}\hat{D}_p^{\mu} - \frac{\imath}{2}\,\hat{\gamma}^{\mu}\,\hat{\gamma}^{\nu}\,\hat{F}_{\mu\nu}(x_p)\;;\quad \hat{F}_{\mu\nu}(x_p) = \hat{t}_{\alpha}\;\hat{F}_{\alpha;\mu\nu}(x_p)\;.$$

In the case of a true vectorial $U_V(1)$ transformation without '$\hat{\gamma}_5$' matrix, the following relations (C.41,C.45) do not apply so that the derived anomaly from the change of the integration measure is caused by the axial '$\hat{\gamma}_5$' property of the transformation. As we insert Eq. (C.38) for the cut-off regulator into (C.37), in order to obtain Eq. (C.39), and as we consider relations (C.40,C.41), we can expand the exponent of (C.39) with the field strength tensor $\hat{F}_{\mu\nu}(x_p)$ up to quadratic order in $(1/M^2)$ (C.42,C.43) as the only remaining term of the Gaussian integrations in the limit $M^2\to\infty$

$$'\mathsf{exp}' = \lim_{M^2\to\infty}\int d^4x_p\frac{d^4k_p}{(2\pi)^4}\;\delta\alpha(x_p)\;\times \tag{C.39}$$
$$\times\underset{N_f,\hat{\gamma}_{mn}^{(\mu)},N_c}{\mathfrak{tr}}\Big[e^{-\imath\,k_p\cdot x_p}\;\hat{\mathfrak{t}}_0\;\hat{\gamma}_5\;\exp\Big\{\Big(\hat{D}_{p,\mu}\hat{D}_P^{\mu} - \frac{\imath}{2}\hat{\gamma}^{\mu}\hat{\gamma}^{\nu}\;\hat{F}_{\mu\nu}(x_p)\Big)\Big/M^2\Big\}\;e^{\imath\,k_p\cdot x_p}\Big]\;;$$
$$f\big(\hat{\partial}_{p,\mu}\big)\;e^{\imath\,k_p\cdot x_p} = e^{\imath\,k_p\cdot x_p}\;f\big(\hat{\partial}_{p,\mu}+\imath\,k_{p,\mu}\big)\;; \tag{C.40}$$

$$\underset{\hat{\gamma}_{mn}^{(\mu)}}{\mathfrak{tr}}\big[\hat{\gamma}_5\big]=0 \quad ; \qquad \underset{\hat{\gamma}_{mn}^{(\mu)}}{\mathfrak{tr}}\big[\hat{\gamma}_5\,\hat{\gamma}_\mu\,\hat{\gamma}_\nu\big]=0\,; \quad \underset{\hat{\gamma}_{mn}^{(\mu)}}{\mathfrak{tr}}\big[\hat{\gamma}_5\,\hat{\gamma}_{\mu_1}\,\ldots\,\hat{\gamma}_{\mu_n}\big]=0 \quad ,\text{(n odd)} \;; \tag{C.41}$$

$$\begin{aligned}
{}'\mathsf{exp}' &= \lim_{M^2\to\infty}\int\frac{d^4k_p}{(2\pi)^4}\;\exp\{-k_{p,\mu}\,k_p^\mu/M^2\}\;\times \\
&\times\;\int d^4x_p\;\delta\alpha(x_p)\underset{N_f,\hat{\gamma}_{mn}^{(\mu)},N_c}{\mathfrak{tr}}\bigg[\hat{\mathfrak{t}}_0\;\hat{\gamma}_5\;\frac{1}{2!}\bigg(-\frac{\imath}{2\,M^2}\hat{\gamma}^\mu\,\hat{\gamma}^\nu\,\hat{F}_{\mu\nu}(x_p)\bigg)\bigg(-\frac{\imath}{2\,M^2}\hat{\gamma}^\kappa\,\hat{\gamma}^\lambda\,\hat{F}_{\kappa\lambda}(x_p)\bigg)\bigg]\;;
\end{aligned}\tag{C.42}$$

$$\begin{aligned}
{}'\mathsf{exp}' &= \lim_{M^2\to\infty}\int\frac{d^4k_p}{(2\pi)^4}\bigg(-\frac{1}{8\,M^4}\;\exp\Big\{-\big(\vec{k}_p\cdot\vec{k}_p-(k_p^0)^2\big)/M^2\Big\}\;\times \\
&\times\;\int d^4x_p\;\delta\alpha(x_p)\underset{\hat{\gamma}_{mn}^{(\mu)}}{\mathfrak{tr}}\Big[\hat{\gamma}_5\;\hat{\gamma}^\mu\;\hat{\gamma}^\nu\;\hat{\gamma}^\kappa\;\hat{\gamma}^\lambda\Big]\underset{N_f,N_c}{\mathfrak{tr}}\Big[\hat{\mathfrak{t}}_0\;\hat{F}_{\mu\nu}(x_p)\;\hat{F}_{\kappa\lambda}(x_p)\Big]\,.
\end{aligned}\tag{C.43}$$

The transformation (C.44) to Euclidean integration variables causes the 'instanton' properties (instead of solitons) of the derived BCS-Hopf invariant in section 5.2 and allows to perform the Gaussian integrations. The total trace in (C.42) splits into a trace of Dirac gamma matrices with the peculiar axial '$\hat{\gamma}_5$' matrix and into a trace of isospin-(flavour-) and colour matrix degrees of freedom (C.43) where we apply the particular trace relation (C.45) of Dirac matrices $\hat{\gamma}^\mu$ with the axial '$\hat{\gamma}_5$' matrix, resulting into the anti-symmetric Levi-Civita symbol (C.45). Using these properties, one finally achieves the chiral anomaly (C.47,C.48) from insertion into the Jacobian (C.34) and into the integration measure (C.33)

$$k_p^0 \;\to\; -\imath\,\omega_p \quad ; \tag{C.44}$$

$$\underset{\hat{\gamma}_{mn}^{(\mu)}}{\mathfrak{tr}}\Big[\hat{\gamma}_5\;\hat{\gamma}^\mu\;\hat{\gamma}^\nu\;\hat{\gamma}^\kappa\;\hat{\gamma}^\lambda\Big] = 4\,\imath\,\varepsilon^{\mu\nu\kappa\lambda}\,;\quad \varepsilon^{0123}=+1 \quad ; \tag{C.45}$$

$$\begin{aligned}
{}'\mathsf{exp}' &= \overbrace{\imath\int\frac{d^3k_p}{(2\pi)^4}\,\frac{d\omega_p}{8\,M^4}\;\exp\Big\{-\big(\vec{k}_p\cdot\vec{k}_p+\omega_p^2\big)/M^2\Big\}}^{(\sqrt{\pi}\,M)^4\cdot(2\pi)^{-4}\cdot(8^{-1}\,M^{-4})=(128\,\pi^2)^{-1}}\;\times \\
&\times\;\int d^4x_p\;\delta\alpha(x_p)\;4\,\imath\,\varepsilon^{\mu\nu\kappa\lambda}\underset{N_f,N_c}{\mathfrak{tr}}\Big[\hat{\mathfrak{t}}_0\;\hat{F}_{\mu\nu}(x_p)\;\hat{F}_{\kappa\lambda}(x_p)\Big]\;;
\end{aligned}\tag{C.46}$$

$${}'\mathsf{exp}' = -\frac{1}{32\,\pi^2}\int d^4x_p\;\delta\alpha(x_p)\;\varepsilon^{\mu\nu\kappa\lambda}\underset{N_f,N_c}{\mathfrak{tr}}\Big[\hat{\mathfrak{t}}_0\;\hat{F}_{\mu\nu}(x_p)\;\hat{F}_{\kappa\lambda}(x_p)\Big]\;; \tag{C.47}$$

$$\begin{aligned}
\prod_{p=\pm}d[\bar{\psi}(x_p)]\;d[\psi(x_p)] &= \prod_{p=\pm}d[\bar{\psi}'(x_p)]\;d[\psi'(x_p)]\;\times \\
&\times\;\exp\bigg\{\frac{\imath}{16\,\pi^2}\int d^4x_p\;\eta_p\;\varepsilon^{\kappa\lambda\mu\nu}\underset{N_f,N_c}{\mathfrak{tr}}\Big[\hat{\mathfrak{t}}_0\;\hat{F}_{\kappa\lambda}(x_p)\;\hat{F}_{\mu\nu}(x_p)\Big]\;\delta\alpha(x_p)\bigg\}\,.
\end{aligned}\tag{C.48}$$

As we combine the transformation of the actions $\mathcal{A}[\psi,\hat{D}_\mu\psi,\hat{F}]$ and $\mathcal{A}_S[\hat{\mathcal{J}},J_\psi,\hat{J}_{\psi\psi},\hat{\mathfrak{j}}^{(\hat{F})}]$ (C.14) in the previous subsection C.1 with the transformation of the integration measure, we accomplish the total axial current relation with chiral anomaly given as the first term on the right-hand side (in the first line of (C.49)) which even remains in the massless limit (last term in the first line of (C.49)). The other terms in lines two to four of Eq. (C.49) follow from the source action $\mathcal{A}_S[\hat{\mathcal{J}},J_\psi,\hat{J}_{\psi\psi},\hat{\mathfrak{j}}^{(\hat{F})}]$ containing the symmetry breaking, (odd-valued) fields $j_{\psi;M}(x_p)$, $j_{\psi;M}^\dagger(x_p)$ and anti-symmetric, (even-valued) matrices $\hat{j}_{\psi\psi;M;N}(x_p)$, $\hat{j}_{\psi\psi;M;N}^\dagger(x_p)$

$$\begin{aligned}
&\boldsymbol{\hat{\partial}_{p,\mu}\Big(\bar{\psi}(x_p)\;\hat{\gamma}^\mu\;\hat{\gamma}_5\;\hat{\mathfrak{t}}_0\;\psi(x_p)\Big)=-\frac{\varepsilon^{\kappa\lambda\mu\nu}}{16\,\pi^2}\underset{N_f,N_c}{\mathfrak{tr}}\Big[\hat{\mathfrak{t}}_0\;\hat{F}_{\kappa\lambda}(x_p)\;\hat{F}_{\mu\nu}(x_p)\Big]+\bar{\psi}(x_p)\;\hat{\gamma}_5\;\big\{\hat{\mathfrak{t}}_0\,,\,\hat{m}\big\}_+\;\psi(x_p)+} \\
&+\quad j_\psi^\dagger(x_p)\;\hat{\gamma}_5\;\hat{\mathfrak{t}}_0\;\psi(x_p)-\psi^\dagger(x_p)\;\hat{\gamma}_5\;\hat{\mathfrak{t}}_0\;j_\psi(x_p)+\frac{1}{2}\bigg[\psi^T(x_p)\Big(\hat{j}_{\psi\psi}^\dagger(x_p)\;\hat{\gamma}_5\;\hat{\mathfrak{t}}_0+\hat{\gamma}_5\;\hat{\mathfrak{t}}_0^T\;\hat{j}_{\psi\psi}^\dagger(x_p)\Big)\psi(x_p)+ \\
&-\quad \psi^\dagger(x_p)\Big(\hat{j}_{\psi\psi}(x_p)\;\hat{\gamma}_5\;\hat{\mathfrak{t}}_0^*+\hat{\gamma}_5\;\hat{\mathfrak{t}}_0\;\hat{j}_{\psi\psi}(x_p)\Big)\psi^*(x_p)\bigg]+\frac{1}{2}\int_C d^4y_q\;\times \\
&\times\quad\bigg[\Psi^\dagger(y_q,x_p)\;\hat{\mathcal{J}}(y_q,x_p)\begin{pmatrix}\hat{\gamma}_5\;\hat{\mathfrak{t}}_0 & \\ & -\hat{\gamma}_5\;\hat{\mathfrak{t}}_0^*\end{pmatrix}\Psi(x_p)+\Psi^\dagger(x_p)\begin{pmatrix}-\hat{\gamma}_5\;\hat{\mathfrak{t}}_0 & \\ & \hat{\gamma}_5\;\hat{\mathfrak{t}}_0^T\end{pmatrix}\hat{\mathcal{J}}(x_p,y_q)\;\Psi(y_q)\bigg]\,.
\end{aligned}\tag{C.49}$$

D Gradient expansion to an effective Lagrangian with nontrivial topology

D.1 Expansion in the anomalous doubled Hilbert space

The original path integral $Z[\hat{\jmath}, J_\psi, \hat{J}_{\psi\psi}, \hat{\mathfrak{j}}^{(\hat{F})}]$ (2.25-2.27) has been changed from matter and gauge fields to self-energies as the appropriate integration variables through subsequent HST's. Furthermore, the path integral (3.63) has been separated by a coset transformation into block diagonal self-energy densities and BCS-terms which are the remaining, most important path field integration variables of the final transformed functional (3.116). Moreover, we have split the gauge field degrees of freedom and scalar quark self-energy densities from the BCS-degrees of freedom by using a background functional averaging with path integral (3.59). The exact expression (3.116) with background functional (3.59) is approximated by factorization of the averaging process (3.59) so that the background functional (3.59) acts with its averaging (D.2) for the potential term $\hat{\mathcal{V}}(x_p)$ (3.60) individually on the actions $\langle\mathcal{A}_{DET}[\hat{T},\hat{\mathcal{V}};\hat{\jmath}]\rangle_{\hat{\mathcal{V}}}$, $\langle\mathcal{A}_{J_\psi}[\hat{T},\hat{\mathcal{V}};\hat{\jmath}]\rangle_{\hat{\mathcal{V}}}$ in the exponents and the BCS-source functional $\langle Z_{\hat{J}_{\psi\psi}}[\hat{T}]\rangle_{\hat{\mathcal{V}}}$

$$Z[\hat{\jmath}, J_\psi, \hat{J}_{\psi\psi}, \hat{\mathfrak{j}}^{(\hat{F})}] \approx \int d[\hat{T}^{-1}(x_p)\, d\hat{T}(x_p)] \;\Big\langle Z_{\hat{J}_{\psi\psi}}[\hat{T}]\Big\rangle_{\hat{\mathcal{V}}} \;\exp\Big\{\Big\langle\mathcal{A}_{DET}[\hat{T},\hat{\mathcal{V}};\hat{\jmath}]\Big\rangle_{\hat{\mathcal{V}}}\Big\}\;\exp\Big\{\imath\Big\langle\mathcal{A}_{J_\psi}[\hat{T},\hat{\mathcal{V}};\hat{\jmath}]\Big\rangle_{\hat{\mathcal{V}}}\Big\}; \tag{D.1}$$

$$\Big\langle\Big(\ldots\Big)\Big\rangle_{\hat{\mathcal{V}}} \;:=\; \Big\langle \boldsymbol{Z\Big[\hat{\mathcal{V}}(x_p);\hat{\mathfrak{S}}^{(\hat{F})}, s_\alpha, \hat{\mathfrak{B}}_{\hat{F}}, \hat{\mathfrak{b}}^{(\hat{F})}, \hat{\mathfrak{U}}_{\hat{F}}, \hat{\mathfrak{v}}_{\hat{F}}; \sigma_D; \hat{\mathfrak{j}}^{(\hat{F})}; \mathbf{Eq.\ (3.59)}\Big]} \times \Big(\ldots\Big)\Big\rangle\,. \tag{D.2}$$

Apart from the derivative- '$\hat{\partial\!\!\!/}_p$', mass- '$\hat{m}$' and '$-\imath\,\hat{\varepsilon}_p$'-terms, the gradient operator $\Delta\hat{\mathcal{H}}_{N;M}^{ba}(y_q,x_p)$ (D.3) in (3.111) $\hat{\mathcal{O}}_{N;M}^{ba}(y_q,x_p)$ consists of the potential $\hat{\mathcal{V}}(x_p)$ (D.4,D.5,3.60) composed of the gauge and quark self-energy density degrees of freedom within the path integral (3.59). The transposition of the '22' part $\hat{H}^T(x_p)$ (D.5) within the anomalous-doubled, one-particle Hamiltonian $\hat{\mathcal{H}}_{N;M}^{ba}(y_q,x_p)$ (D.3) involves the internal symmetry spaces as the isopspin-(flavour-) matrices, the 4×4 Dirac gamma matrices and the gauge field generators $\hat{t}_\alpha$, and also the contour spacetime derivative '$(\hat{\partial}_{p,\mu})^T = -\hat{\partial}_{p,\mu}$' which results into an additional minus sign. Corresponding to notations and definitions of gamma-matrices [20], one obtains for the gradient operator $\Delta\hat{\mathcal{H}}_{N;M}^{ba}(y_q,x_p)$ with the coset matrices $\hat{T}^{-1}(x_p)$, $\hat{T}(x_p)$ for the remaining BCS-degrees of freedom following equations

$$\Delta\hat{\mathcal{H}}_{N;M}^{ba}(y_q,x_p) = \Big(\hat{T}^{-1}\hat{\mathcal{H}}\hat{T} - \hat{\mathcal{H}}\Big)_{N;M}^{ba}(y_q,x_p) = \delta^{(4)}(y_q-x_p)\;\eta_q\;\delta_{qp}\times \tag{D.3}$$
$$\times\;\left[\hat{T}_{N;N'}^{-1;bb'}(x_p)\begin{pmatrix}\hat{H}_{N';M'}(x_p) & 0\\ 0 & \hat{H}_{N';M'}^T(x_p)\end{pmatrix}_{N';M'}^{b'a'}\hat{T}_{M';M}^{a'a}(x_p) - \begin{pmatrix}\hat{H}_{N;M}(x_p) & 0\\ 0 & \hat{H}_{N;M}^T(x_p)\end{pmatrix}_{N;M}^{ba}\right];$$

$$\begin{aligned}\hat{H}(x_p) &= \Big[\hat{\beta}\Big(\hat{\partial\!\!\!/}_p + \imath\,\hat{\mathcal{V}}(x_p) - \imath\,\hat{\varepsilon}_p + \hat{m}\Big)\Big]\,; \quad \Big(\hat{\varepsilon}_p = \hat{\beta}\,\varepsilon_p = \hat{\beta}\,\eta_p\,\varepsilon_+\,;\quad \varepsilon_+ > 0\Big)\,; & \text{(D.4)}\\ &= \hat{\beta}\,\hat{\gamma}^\mu\,\hat{\partial}_{p,\mu} - \imath\,\varepsilon_p\,\hat{1}_{N_0\times N_0} + \hat{\beta}\,\big(\imath\,\hat{\mathcal{V}}(x_p) + \hat{m}\big)\,;\\ \hat{H}^T(x_p) &= \big(\hat{\beta}\,\hat{\gamma}^\mu\big)^T(-\hat{\partial}_{p,\mu}) - \imath\,\varepsilon_p\,\hat{1}_{N_0\times N_0} + \big[\hat{\beta}\,\big(\imath\,\hat{\mathcal{V}}(x_p)+\hat{m}\big)\big]^T\,. & \text{(D.5)}\end{aligned}$$

As the gradient operators $\hat{\beta}\,\hat{\partial\!\!\!/}_p$ and its transposes $(\hat{\beta}\,\hat{\partial\!\!\!/}_p\,)^T$ act onto the coset matrices $\hat{T}_{M';M}^{a'a}(x_p)$, $\hat{T}_{N;N'}^{-1;bb'}(x_p)$ to the right or left, it has to be distinguished between saturated derivatives as $(\hat{\partial}_{p,\mu}\hat{T}(x_p)\,)$ whose actions are limited by outer braces (in the given case for a single coset matrix) and unsaturated derivative 'operators' acting beyond the outer coset matrices further to the right and left onto other terms in the gradient expansion. Therefore, we consider two parts $\delta\hat{\mathfrak{h}}(x_p) = \hat{\mathfrak{k}}(x_p) + \delta\hat{\mathfrak{V}}(x_p)$ (D.7,D.8) and $\delta\hat{\mathcal{K}}^{\boldsymbol{\mu}}(x_p)\;\boldsymbol{\hat{\partial}_{p,\mu}}$ (D.9) of the total gradient operator $\Delta\hat{\mathcal{H}}$ (D.3-D.6) which consists of 'saturated' derivatives of coset matrices $(\hat{\partial}_{p,\mu}\hat{T}(x_p)\,)$ with additional potential matrix $\hat{\mathfrak{V}}(x_p)$ and 'unsaturated' gradient '*operators*' $\boldsymbol{\hat{\partial}_{p,\mu}}$ with Hamiltonian $\delta\hat{\mathcal{K}}^{\boldsymbol{\mu}}(x_p)$ (D.9), respectively. This difference is symbolized by boldface letters for the 'unsaturated' gradient '*operator*' $\boldsymbol{\hat{\partial}_{p,\mu}}$ in the following expansions. We also abbreviate the block diagonal anomalous-doubled potential matrix of the background potential $\mathcal{V}_\beta^\mu(x_p)$ (3.60) and the isospin- (flavour-) masses $\hat{m}$, $\hat{m}^T$ by the additional symbol $\hat{\mathfrak{V}}_{N;M}^{ba}(x_p)$ (D.8). This anomalous-doubled potential $\hat{\mathfrak{V}}_{N;M}^{ba}(x_p)$ (D.8) is only contained in the

Hamiltonian part $\delta\hat{\mathfrak{h}}(x_p)$ (D.7) with saturated gradients and gives rise to effective coupling functions $\langle\mathcal{V}_\beta^\mu(x_p)\;\mathcal{V}_\gamma^\nu(x_p)\rangle_{\hat{\mathcal{V}}}$ from the averaging with the background functional (3.59,D.2). Note that the potential matrix $\hat{\mathfrak{V}}(x_p)=\mathcal{V}_{\alpha;\mu}(x_p)\;\hat{V}_\alpha^\mu$ of background field $\mathcal{V}_{\alpha;\mu}(x_p)$ is weighted by the coset matrices $\hat{T}^{-1}(x_p)$, $\hat{T}(x_p)$ so that one achieves the defined difference $\delta\hat{\mathfrak{V}}(x_p)=\hat{T}^{-1}(x_p)\;\hat{\mathfrak{V}}(x_p)\;\hat{T}(x_p)-\hat{\mathfrak{V}}(x_p)=\mathcal{V}_{\alpha;\mu}(x_p)\;(\hat{T}^{-1}(x_p)\;\hat{V}_\alpha^\mu\;\hat{T}(x_p)-\hat{V}_\alpha^\mu)$ because of the original gradient term $\Delta\hat{\mathcal{H}}=(\hat{T}^{-1}\,\hat{\mathcal{H}}\,\hat{T}-\hat{\mathcal{H}})$ and because of the non-commuting property $[\hat{T}(x_p)\,,\,\hat{\mathfrak{V}}(x_p)\,]_-\neq 0$

$$\begin{aligned}
\Delta\hat{\mathcal{H}}_{N;M}^{ba}(y_q,x_p) &= \delta^{(4)}(y_q-x_p)\;\eta_q\;\delta_{qp}\Big[\delta\hat{\mathfrak{h}}_{N;M}^{ba}(x_p)+\delta\hat{\mathcal{K}}_{N;M}^{\boldsymbol{\mu};ba}(x_p)\;\hat{\boldsymbol{\partial}}_{\boldsymbol{p},\boldsymbol{\mu}}\Big] && \text{(D.6)}\\
&= \delta^{(4)}(y_q-x_p)\;\eta_q\;\delta_{qp}\Big[\hat{\mathfrak{k}}_{N;M}^{ba}(x_p)+\delta\hat{\mathfrak{V}}_{N;M}^{ba}(x_p)+\delta\hat{\mathcal{K}}_{N;M}^{\boldsymbol{\mu};ba}(x_p)\;\hat{\boldsymbol{\partial}}_{\boldsymbol{p},\boldsymbol{\mu}}\Big]\;;\\
\delta\hat{\mathfrak{h}}_{N;M}^{ba}(x_p) &= \hat{\mathfrak{k}}_{N;M}^{ba}(x_p)+\delta\hat{\mathfrak{V}}_{N;M}^{ba}(x_p)\;;\;\;\to\hat{\mathfrak{k}}_{N;M}^{ba}(x_p)=\Big[\hat{T}^{-1}(x_p)\;\hat{\beta}\;\hat{\gamma}^\mu\;\hat{S}\;\big(\hat{\partial}_{p,\mu}\hat{T}(x_p)\big)\Big]_{N;M}^{ba}\;; && \text{(D.7)}\\
\delta\hat{\mathfrak{V}}_{N;M}^{ba}(x_p) &= \Big[\hat{T}^{-1}(x_p)\,\hat{\mathfrak{V}}(x_p)\,\hat{T}(x_p)-\hat{\mathfrak{V}}(x_p)\Big]_{N;M}^{ba}=\mathcal{V}_{\alpha;\mu}(x_p)\;\Big[\hat{T}^{-1}(x_p)\;\hat{V}_\alpha^\mu\;\hat{T}(x_p)-\hat{V}_\alpha^\mu\Big]_{N;M}^{ba}; && \text{(D.8)}\\
\big[\hat{V}_\alpha^\mu\big]_{N;M}^{ba} &= \hat{V}_{\alpha;N;M}^{\mu;ba}=\begin{pmatrix}\big[\hat{\beta}\,(\imath\,\hat{\gamma}^\mu\,\hat{t}_\alpha+\hat{m})\big]_{N;M} & 0\\ 0 & \big[\hat{\beta}\,(\imath\,\hat{\gamma}^\mu\,\hat{t}_\alpha+\hat{m})\big]_{N;M}^T\end{pmatrix}^{ba}\delta_{ab}\;;\\
\hat{\mathfrak{V}}_{N;M}^{ba}(x_p) &= \delta_{ab}\begin{pmatrix}\big[\hat{\beta}\,(\imath\,\hat{\mathcal{V}}(x_p)+\hat{m})\big]_{N;M} & 0\\ 0 & \big[\hat{\beta}\,(\imath\,\hat{\mathcal{V}}(x_p)+\hat{m})\big]_{N;M}^T\end{pmatrix}^{ba}\\
&= \mathcal{V}_{\alpha;\mu}(x_p)\begin{pmatrix}\big[\hat{\beta}\,(\imath\,\hat{\gamma}^\mu\,\hat{t}_\alpha+\hat{m})\big]_{N;M} & 0\\ 0 & \big[\hat{\beta}\,(\imath\,\hat{\gamma}^\mu\,\hat{t}_\alpha+\hat{m})\big]_{N;M}^T\end{pmatrix}^{ba}=\mathcal{V}_{\alpha;\mu}(x_p)\;\hat{V}_{\alpha;N;M}^{\mu;ba}\;;\\
\delta\hat{\mathcal{K}}_{N;M}^{\boldsymbol{\mu};ba}(x_p) &= \Big(\hat{T}^{-1}(x_p)\;\hat{\beta}\;\hat{\gamma}^{\boldsymbol{\mu}}\;\hat{S}\;\hat{T}(x_p)\Big)_{N;M}^{ba}-\delta_{ab}\;\Big(\hat{\beta}\;\hat{\gamma}^{\boldsymbol{\mu}}\;\hat{S}\Big)_{N;M}^{ba}\,. && \text{(D.9)}
\end{aligned}$$

Aside from the background potential matrix $\hat{\mathfrak{V}}(x_p)$ (D.8), the anomalous-doubled one-particle operator $\hat{\mathcal{H}}(x_p)$ (D.10) has the diagonal '$-\imath\,\varepsilon_p\;\hat{1}_{2N_0\times 2N_0}$' non-hermitian part which determines the analytic behaviour of Green functions $\hat{G}^{(0)}$, $\hat{g}^{(0)}$, $[\hat{g}^{(0)}]^T$ propagating on the non-equilibrium time contour. The anomalous-doubled Green function $\hat{G}^{(0)}$ (D.11), consisting of the block diagonal parts $\hat{g}^{(0)}$, $[\hat{g}^{(0)}]^T$ (D.12,D.13), differ from the simple inverses $\hat{\mathcal{H}}^{-1}$, $[\hat{H}(x_p)]^{-1}$, $[\hat{H}^T(x_p)]^{-1}$ of the operators (D.4,D.5) by the additional averaging with the background functional (3.59,D.2) for the potential $\hat{\mathcal{V}}(x_p)$ (3.60). This averaging procedure for products of several inverse operators with $\hat{\mathcal{H}}^{-1}$ is simplified to corresponding products with independent background averaging of single operators $\hat{\mathcal{H}}^{-1}$ leading to factors of proper non-equilibrium Green functions. This means that we neglect any correlations between the potentials $\mathcal{V}_\beta^\mu(x_p)$ (3.60), $\hat{\mathfrak{V}}(x_p)$ (D.8) in the factors of inverses $\langle\hat{\mathcal{H}}^{-1}{}_\times\ldots{}_\times\hat{\mathcal{H}}^{-1}\rangle_{\hat{\mathcal{V}}}$ and simply reduce to $\langle\hat{\mathcal{H}}^{-1}\rangle_{\hat{\mathcal{V}}}{}_\times\ldots{}_\times\langle\hat{\mathcal{H}}^{-1}\rangle_{\hat{\mathcal{V}}}=\hat{G}^{(0)}{}_\times\ldots{}_\times\hat{G}^{(0)}$

$$\begin{aligned}
\hat{\mathcal{H}}(x_p) &= \hat{\beta}\;\hat{\gamma}^\mu\;\hat{S}\;\hat{\boldsymbol{\partial}}_{\boldsymbol{p},\boldsymbol{\mu}}+\hat{\mathfrak{V}}_{N;M}^{ba}(x_p)-\imath\,\varepsilon_p\;\hat{1}_{2N_0\times 2N_0}\;; && \text{(D.10)}\\
\hat{G}^{(0)} &= \Big\langle\hat{\mathcal{H}}^{-1}\Big\rangle_{\hat{\mathcal{V}}}=\begin{pmatrix}\hat{g}^{(0)} & 0\\ 0 & \big[\hat{g}^{(0)}\big]^T\end{pmatrix}; && \text{(D.11)}\\
\hat{g}_{g,n,s;f,m,r}^{(0)} &= \Big\langle\big[\hat{H}(x_p)\big]_{N;M}^{-1}\Big\rangle_{\hat{\mathcal{V}}}=\Big\langle\big[\hat{\beta}\big(\hat{\partial\!\!\!/}_p+\imath\,\hat{\mathcal{V}}(x_p)+\hat{m}\big)-\imath\,\varepsilon_p\big]_{N;M}^{-1}\Big\rangle_{\hat{\mathcal{V}}} && \text{(D.12)}\\
&\approx \Big[\hat{\beta}\Big(\hat{\partial\!\!\!/}_p+\imath\,\langle\hat{\mathcal{V}}(x_p)\rangle_{\hat{\mathcal{V}}}+\hat{m}\Big)-\imath\,\varepsilon_p\Big]_{N;M}^{-1}\;;\\
\big[\hat{g}^{(0)}\big]_{g,n,s;f,m,r}^T &= \Big\langle\big[\hat{H}^T(x_p)\big]_{N;M}^{-1}\Big\rangle_{\hat{\mathcal{V}}}=\Big\langle\big[\hat{\beta}\big(\hat{\partial\!\!\!/}_p+\imath\,\hat{\mathcal{V}}(x_p)+\hat{m}\big)-\imath\,\varepsilon_p\big]_{N;M}^{T;-1}\Big\rangle_{\hat{\mathcal{V}}} && \text{(D.13)}\\
&\approx \Big[\hat{\beta}\Big(\hat{\partial\!\!\!/}_p+\imath\,\langle\hat{\mathcal{V}}(x_p)\rangle_{\hat{\mathcal{V}}}+\hat{m}\Big)-\imath\,\varepsilon_p\Big]_{N;M}^{T;-1}\,.
\end{aligned}$$

Proceeding as in chapter 4 of [11], one has to determine the anomalous-doubled, averaged time contour Green function $\hat{G}^{(0)} = \langle\hat{\mathcal{H}}^{-1}\rangle_{\hat{\mathcal{V}}}$ (D.11-D.13) with the transpose of the '11' block extended to the '22' block. This can be accomplished by a saddle point approximation of the background functional (3.59) with its various gauge field variables and quark self-energy densities. However, the imaginary parts of $\hat{\beta}\,\hat{\mathcal{V}}(x_p)$ (3.60) or $\hat{\mathfrak{V}}(x_p)$ (D.8), resulting from a saddle point approximation, have to comply with the imaginary contour sign '$-\imath\,\varepsilon_p$' for a stable propagation of coset matrices. We use the overview of the anomalous-doubled Hilbert space, summarized in appendix A, and the definitions and notations of chapter 4 in [11] so that the 'anti-unitary', 'anti-linear' '22' states accompany as extensions the original states in the '11' block. It has to be taken into account that the square root of the determinant follows from integration over the bilinear anti-commuting fields $\psi_M(x_p)$ which are doubled by their complex conjugates $\psi^*_M(x_p)$. Consequently a Hilbert space for $\psi_M(x_p)$ with 'ket' $|\psi_M\rangle$ has also to be doubled by its 'dual' space $\overline{|\psi_M\rangle} = \langle\psi_M|$ the 'bra'. The corresponding Hilbert space of spacetime variables has therefore also to comprise the anti-linear part $\overline{|x_p\rangle} = \langle x_p|$ in its section $a=2$

$$\widehat{|x_p\rangle}^{a(=1,2)} = \begin{pmatrix} |x_p\rangle^{a=1} \\ \overline{|x_p\rangle}^{a=2} \end{pmatrix} = \begin{pmatrix} |x_p\rangle \\ \langle x_p| \end{pmatrix}^{a(=1,2)} . \tag{D.14}$$

The application of rules for an anomalous-doubled Hilbert space, defined in appendix A, leads to the background averaged matrix representations of Green functions in the '11' part and its transposed '22' part with the particular anti-linear $\overline{\langle x_p|}$, $\overline{|y_q\rangle}$ spacetime states (D.15-D.19). According to general properties of contour time Green functions, one obtains generalized time contour Heaviside functions $\theta_{pq}(x^0-y^0)$ and background averaged time development operators $\langle\,\langle\vec{x}|\hat{\mathfrak{U}}(x^0,y^0)|\vec{y}\rangle\,\rangle_{\hat{\mathcal{V}}}$. The matrix representation of the latter time development operator is constructed from standard time path ordering $x^0 > z^0 > y^0$, denoted by the arrow $\overleftarrow{\exp}$, whereas the generalized Heaviside function $\theta_{pq}(x^0-y^0)$ is relevant for the additional appropriate ordering on the '*contour extended*' times. Therefore, one always obtains an inverse propagation of the '22' block relative to the '11' block concerning contour extended times of the Heaviside functions $\theta_{pq}(x^0-y^0)$ within the anomalous-doubled Green functions (D.17)

$$\begin{aligned} \langle x_p|\hat{g}^{(0)}|y_q\rangle &= \Big\langle \langle x_p|\big[\hat{\beta}(\hat{\partial\!\!\!/}_p + \imath\hat{\mathcal{V}}(x_p) + \hat{m}) - \imath\,\varepsilon_p\big]^{-1}|y_q\rangle\Big\rangle_{\hat{\mathcal{V}}} \qquad (D.15)\\ &= \imath\,\theta_{pq}(x^0-y^0)\;\Big\langle\langle\vec{x}|\hat{\mathfrak{U}}(x^0,y^0)\,|\vec{y}\rangle\Big\rangle_{\hat{\mathcal{V}}}\;; \end{aligned}$$

$$\overline{\langle x_p|}\big[\hat{g}^{(0)}\big]^T\overline{|y_q\rangle} = \langle y_q|\hat{g}^{(0)}|x_p\rangle = \imath\,\theta_{qp}(y^0-x^0)\;\Big\langle\langle\vec{y}|\hat{\mathfrak{U}}(y^0,x^0)\,|\vec{x}\rangle\Big\rangle_{\hat{\mathcal{V}}}\;; \tag{D.16}$$

$${}^{a}\widehat{\langle x_p|}\hat{G}^{(0)}\widehat{|y_q\rangle}^{b} = \delta_{ab}\,\imath\begin{pmatrix} \theta_{pq}(x^0-y^0)\;\Big\langle\langle\vec{x}|\hat{\mathfrak{U}}(x^0,y^0)\,|\vec{y}\rangle\Big\rangle_{\hat{\mathcal{V}}} & 0 \\ 0 & \theta_{qp}(y^0-x^0)\;\Big\langle\langle\vec{y}|\hat{\mathfrak{U}}(y^0,x^0)\,|\vec{x}\rangle\Big\rangle_{\hat{\mathcal{V}}} \end{pmatrix}; \tag{D.17}$$

$$\begin{aligned} \Big\langle\langle\vec{x}|\hat{\mathfrak{U}}(x^0,y^0)\,|\vec{y}\rangle\Big\rangle_{\hat{\mathcal{V}}} &= \Big\langle \overleftarrow{\exp}\Big\{-\imath\int_{y^0}^{x^0}dz^0\;\Big[\hat{\beta}\big(\vec{\gamma}\cdot\vec{\partial}_{\vec{x}} + \imath\hat{\mathcal{V}}(z^0,\vec{x}) + \hat{m}\big) - \imath\,\varepsilon_+\Big]\Big\}\delta^{(3)}(\vec{x}-\vec{y})\Big\rangle_{\hat{\mathcal{V}}} \qquad (D.18)\\ &\approx \overleftarrow{\exp}\Big\{-\imath\int_{y^0}^{x^0}dz^0\;\Big[\hat{\beta}\big(\vec{\gamma}\cdot\vec{\partial}_{\vec{x}} + \imath\,\langle\hat{\mathcal{V}}(z^0,\vec{x})\rangle_{\hat{\mathcal{V}}} + \hat{m}\big) - \imath\,\varepsilon_+\Big]\Big\}\delta^{(3)}(\vec{x}-\vec{y})\;; \end{aligned}$$

$$\begin{aligned} \delta(x^0-y^0)\;\delta^{(3)}(\vec{x}-\vec{y}) &= \Big\langle\Big[\hat{\beta}\big(\hat{\gamma}^0\,\hat{\partial}_{x^0} + \vec{\gamma}\cdot\vec{\partial}_{\vec{x}} + \imath\hat{\mathcal{V}}(x^0,\vec{x}) + \hat{m}\big)\Big]\qquad\langle\vec{x}|\hat{\mathfrak{U}}(x^0,y^0)\,|\vec{y}\rangle\Big\rangle_{\hat{\mathcal{V}}} \qquad (D.19)\\ &\approx \Big[\hat{\beta}\big(\hat{\gamma}^0\,\hat{\partial}_{x^0} + \vec{\gamma}\cdot\vec{\partial}_{\vec{x}} + \imath\,\Big\langle\hat{\mathcal{V}}(x^0,\vec{x})\Big\rangle_{\hat{\mathcal{V}}} + \hat{m}\big)\Big]\qquad\Big\langle\langle\vec{x}|\hat{\mathfrak{U}}(x^0,y^0)\,|\vec{y}\rangle\Big\rangle_{\hat{\mathcal{V}}}\;. \end{aligned}$$

We specify in relations (D.20,D.21) the precise from of contour extended Heaviside functions $\theta_{pq}(x^0-y^0)$ in terms of the standard Heaviside function $\theta(x^0-y^0)$ and conclude from the standard relation between contour extended Heaviside functions the particular expression (D.21) between anomalous-doubled, block diagonal time contour Green functions. The equation (D.21) also requires a block diagonal, anomalous-doubled Heaviside function $\Theta_{pq}^{ab}(x^0-y^0)$ which we define in (D.22-D.24)

$$\theta_{p=+,q=+}(x^0-y^0) = \theta(x^0-y^0) = +1 \quad\text{for}\quad x^0 \geq y^0\;;$$

$$\begin{aligned}
\theta_{p=-,q=-}(x^0-y^0) &= \theta(y^0-x^0)=+1 \quad \text{for}\;\; y^0\geq x^0\;; \\
\theta_{p=-,q=+}(x^0-y^0) &\equiv 1 \quad \text{for all}\;\; x^0,\;y^0\;; \\
\theta_{p=+,q=-}(x^0-y^0) &\equiv 0 \quad \text{for all}\;\; x^0,\;y^0\;; \\
\theta(x^0-y^0) &= +1 \;\; (\text{for}\;\; x^0\geq y^0) \quad \text{and} \quad =0 \;\; (\text{for}\;\; x^0<y^0)\;; \\
\theta_{++}(x^0-y^0)+\theta_{--}(x^0-y^0+0_+) &= \underbrace{\theta_{-+}(x^0-y^0)}_{\equiv 1}+\underbrace{\theta_{+-}(x^0-y^0)}_{\equiv 0}\;;
\end{aligned} \tag{D.20}$$

$$\;^{a}\widehat{\langle x_+|}\hat{G}^{(0)}\widehat{|y_+\rangle}^{a}+\;^{a}\widehat{\langle x_-|}\hat{G}^{(0)}\widehat{|y_-\rangle}^{a} = \underbrace{\;^{a}\widehat{\langle x_-|}\hat{G}^{(0)}\widehat{|y_+\rangle}^{a}}_{\neq 0 \text{ for } a=1\text{ ,but } \equiv 0 \text{ for } a=2} + \underbrace{\;^{a}\widehat{\langle x_+|}\hat{G}^{(0)}\widehat{|y_-\rangle}^{a}}_{\equiv 0 \text{ for } a=1\text{ ,but } \neq 0 \text{ for } a=2}\;; \tag{D.21}$$

$$\widehat{\;^{a}\langle x_p|}\hat{G}^{(0)}\widehat{|y_q\rangle}^{b} = \delta_{ab}\;\imath\;\Theta_{pq}^{ab}(x^0-y^0)\begin{pmatrix} \Big\langle\langle\vec{x}|\hat{\mathfrak{U}}(x^0,y^0)\,|\vec{y}\rangle\Big\rangle_{\hat{\mathcal{V}}} & 0 \\ 0 & \Big\langle\langle\vec{y}|\hat{\mathfrak{U}}(y^0,x^0)\,|\vec{x}\rangle\Big\rangle_{\hat{\mathcal{V}}}\end{pmatrix}; \tag{D.22}$$

$$\Theta_{pq}^{ab}(x^0-y^0) = \delta_{ab}\begin{pmatrix}\theta_{pq}(x^0-y^0) & 0 \\ 0 & \theta_{qp}(y^0-x^0)\end{pmatrix}^{ab}; \tag{D.23}$$

$$\Theta_{pp}^{ab}(x^0-y^0) = \delta_{ab}\begin{pmatrix}\theta_{pp}(x^0-y^0) & 0 \\ 0 & \theta_{pp}(y^0-x^0)\end{pmatrix} = \delta_{ab}\begin{cases}\begin{pmatrix}\theta(x^0-y^0) & 0 \\ 0 & \theta(y^0-x^0)\end{pmatrix} & \text{for}\;\; p=+ \\ \begin{pmatrix}\theta(y^0-x^0) & 0 \\ 0 & \theta(x^0-y^0)\end{pmatrix} & \text{for}\;\; p=-\end{cases}. \tag{D.24}$$

The anomalous-doubled Heaviside or time contour step function $\Theta_{pq}^{ab}(x^0-y^0)$ restricts the possible terms in the gradient expansion of the action $\langle\mathcal{A}_{DET}[\hat{T},\hat{\mathcal{V}};\hat{\partial}]\rangle_{\hat{\mathcal{V}}}$ because the trace operations also involve the contour extended traces of spacetime; in consequence one attains as the remaining terms in the gradient expansion of $\langle\mathcal{A}_{DET}[\hat{T},\hat{\mathcal{V}};\hat{\partial}]\rangle_{\hat{\mathcal{V}}}$ only those in which the anomalous-doubled time contour step functions $\Theta_{pp}^{ab}(x^0-y^0)$ do not result into contradictory propagations concerning the time contour extended ordering of gradient terms $\Delta\hat{\mathfrak{H}}(x_p)$ (e.g. $\Theta_{pq}^{aa}(x^0-y^0)\;\Theta_{qp}^{aa}(y^0-x^0)=0\;!$). This restriction of terms is missing in the case of the gradient expansion of $\langle\mathcal{A}_{J_\psi}[\hat{T},\hat{\mathcal{V}};\hat{\partial}]\rangle_{\hat{\mathcal{V}}}$. In continuation of principles for a gradient expansion, we state that an anomalous-doubled field $\Psi_M^a(x_p)$ propagates with the block diagonal, doubled Green function $\;^{a}\widehat{\langle x_p|}\hat{G}^{(0)}\widehat{|y_q\rangle}^{b}$ (D.25) with background potential $\mathcal{V}_\beta^\mu(x_p)$ (3.60). This principle has to be used in the expansion of $\langle\mathcal{A}_{J_\psi}[\hat{T},\hat{\mathcal{V}};\hat{\partial}]\rangle_{\hat{\mathcal{V}}}$ where one starts to propagate with the source field $J_{\psi;M}^a(x_p)$ on the right-hand side of the action for a coherent wavefunction. It replaces the wavefunction $\Psi_M^a(x_p)$ in (D.25). We generalize rule (D.25) for the propagation of arbitrary fields $(\,\mathfrak{w}_M(x_p)\,;\,\mathfrak{w}_M^*(x_p)\,)^{T,a}$ and list in Eqs. (D.26-D.28) also the propagation for the split parts $\mathfrak{w}_M(x_p)$ and its complex conjugated field $\mathfrak{w}_M^*(x_p)$. The doubled fields $(\,\mathfrak{w}_M(x_p)\,;\,\mathfrak{w}_M^*(x_p)\,)^{T,a}$ can be identified with the fields in the various steps of propagation in $\langle\mathcal{A}_{J_\psi}[\hat{T},\hat{\mathcal{V}};\hat{\partial}]\rangle_{\hat{\mathcal{V}}}$ with $\hat{G}^{(0)}$, starting from $J_{\psi;M}^a(x_p)$ on the right-hand side

$$\Psi_M^{a(=1,2)}(x_p) = \begin{pmatrix}\psi_M(x_p) \\ \psi_M^*(x_p)\end{pmatrix}^a = \int_C d^4y_q\;\mathcal{N}^2\;\;^{a}\widehat{\langle x_p|}\hat{G}_{M;N}^{(0)}\widehat{|y_q\rangle}^{b}\begin{pmatrix}\psi_N(y_q) \\ \psi_N^*(y_q)\end{pmatrix}^b; \tag{D.25}$$

$$\mathfrak{w}_M(x_p) = \int_C d^4y_q\;\mathcal{N}^2\;\langle x_p|\hat{g}_{M;N}^{(0)}|y_q\rangle\;\mathfrak{w}_N(y_q)\;; \tag{D.26}$$

$$\mathfrak{w}_M^*(x_p) = \int_C d^4y_q\;\mathcal{N}^2\;\overline{\langle x_p|}\big[\hat{g}\big]_{M;N}^{(0)}\overline{|y_q\rangle}\;\mathfrak{w}_N^*(y_q) = \int_C d^4y_q\;\mathcal{N}^2\;\mathfrak{w}_N^*(y_q)\;\langle y_q|\hat{g}_{N;M}^{(0)}|x_p\rangle\;; \tag{D.27}$$

$$\begin{pmatrix}\mathfrak{w}_M(x_p) \\ \mathfrak{w}_M^*(x_p)\end{pmatrix}^a = \int_C d^4y_q\;\mathcal{N}^2\;\;^{a}\widehat{\langle x_p|}\hat{G}_{M;N}^{(0)}\widehat{|y_q\rangle}^{b}\begin{pmatrix}\mathfrak{w}_N(y_q) \\ \mathfrak{w}_N^*(y_q)\end{pmatrix}^b. \tag{D.28}$$

However, the propagation of fields with relations (D.25-D.28) is not directly applicable for the action $\langle\mathcal{A}_{DET}[\hat{T},\hat{\mathcal{V}};\hat{\partial}]\rangle_{\hat{\mathcal{V}}}$ because of the cyclic invariance of traces, both of the internal state space and the Hilbert space trace of doubled quantum

mechanics. The Hilbert space trace means a propagation back to the same spacetime point. This property is not included in rules (D.25-D.28) which can only be used directly for the source field $J_{\psi;M}^{a}(x_p)$ (as a 'condensate seed') with repeated propagation of $\hat{G}^{(0)}$ to the left-hand side $J_{\psi;N}^{\dagger,b}(y_q)$ in the action $\langle\mathcal{A}_{J_\psi}[\hat{T},\hat{\mathcal{V}};\hat{\mathfrak{J}}]\rangle_{\hat{\mathcal{V}}}$. We circumvent this problem by introducing anomalous-doubled unit operators of momentum-energy states in the expansion of $\langle\mathcal{A}_{DET}[\hat{T},\hat{\mathcal{V}};\hat{\mathfrak{J}}]\rangle_{\hat{\mathcal{V}}}$ so that one achieves anomalous-doubled plane wavefunctions with definite momentum-energy values instead of the source fields $J_{\psi;N}^{\dagger,b}(y_q)$, $J_{\psi;M}^{a}(x_p)$. The anomalous doubled plane wave states replace the fields in (D.25-D.28) and propagate with the time contour extended, background averaged Green functions (D.11-D.24) from the right-hand side to the left-hand side or vice versa. Since the various gradient terms follow straightforwardly for $\langle\mathcal{A}_{J_\psi}[\hat{T},\hat{\mathcal{V}};\hat{\mathfrak{J}}]\rangle_{\hat{\mathcal{V}}}$ by repeated application of rules (D.25-D.28) and gradient parts $\delta\mathfrak{h}(x_p)$, $\delta\hat{\mathcal{K}}^{\boldsymbol{\mu}}(x_p)\ \hat{\boldsymbol{\partial}}_{\boldsymbol{p},\boldsymbol{\mu}}$ (D.3-D.9), we give details of the expansion for the more involved problem of traces in $\langle\mathcal{A}_{DET}[\hat{T},\hat{\mathcal{V}};\hat{\mathfrak{J}}]\rangle_{\hat{\mathcal{V}}}$. The gradient expansion of $\langle\mathcal{A}_{J_\psi}[\hat{T},\hat{\mathcal{V}};\hat{\mathfrak{J}}]\rangle_{\hat{\mathcal{V}}}$ can then be obtained from that of $\langle\mathcal{A}_{DET}[\hat{T},\hat{\mathcal{V}};\hat{\mathfrak{J}}]\rangle_{\hat{\mathcal{V}}}$ by replacing anomalous-doubled plane wave states through the anomalous-doubled source fields $J_{\psi;N}^{\dagger,b}(y_q)$, $J_{\psi;M}^{a}(x_p)$ which have simpler propagation terms without the projection matrix $\hat{S}$ for the coset space. Therefore, we concentrate in the following subsection D.2 onto the gradient expansion of the action $\langle\mathcal{A}_{DET}[\hat{T},\hat{\mathcal{V}};\hat{\mathfrak{J}}]\rangle_{\hat{\mathcal{V}}}$ and summarize the effective Lagrangian up to the complete fourth order gradient with supplementary terms of $\langle\mathcal{A}_{J_\psi}[\hat{T},\hat{\mathcal{V}};\hat{\mathfrak{J}}]\rangle_{\hat{\mathcal{V}}}$ in part D.3.

D.2 Gradient expansion of the anomalous-doubled determinant

In order to allow for stable, static energy configurations in 3+1 spacetime dimensions, one has to expand up to fourth order gradients so that one cannot scale the particular configuration to arbitrary small or large sizes in the three dimensional coordinate space integrations over the static Hamiltonian density ('Derrick's theorem' [13, 15] !). Since we have shifted the trace-logarithm (3.57-3.59) of the one-particle operator $\hat{\mathcal{H}}(x_p)$ with anomalous doubled potential $\mathcal{V}_{\beta}^{\mu}(x_p)$, $\hat{\mathfrak{V}}(x_p)=\mathcal{V}_{\beta;\mu}(x_p)\cdot\hat{V}_{\beta}^{\mu}$ (3.60,D.8) for quark self-energy densities and gauge degrees of freedom to the background functional (3.59), the trace-logarithm terms $\langle$'traces' $\ln[\hat{\mathcal{H}}]\rangle_{\hat{\mathcal{V}}}$ cancel in the expansion of $\langle\mathcal{A}_{DET}[\hat{T},\hat{\mathcal{V}};\hat{\mathfrak{J}}]\rangle_{\hat{\mathcal{V}}}$ which is specified in (D.29). The first order source term with $\widetilde{\mathfrak{J}}(\hat{T}^{-1},\hat{T})$ (D.30) is listed in detail in (D.29) with remaining unspecified terms ascending from second order $(\,\widetilde{\mathfrak{J}}(\hat{T}^{-1},\hat{T})\,)^{n\geq 2}$. This source term $\widetilde{\mathfrak{J}}(\hat{T}^{-1},\hat{T})$ (D.29,D.30) allows to track the form of observables from the original anti-commuting Fermi fields to those in terms of the complex-, even-valued self-energy matrix (3.64-3.83). However, we concentrate in this subsection on the derivation of the final effective Lagrangian and set the important source term $\widetilde{\mathfrak{J}}(\hat{T}^{-1},\hat{T})$ to zero in further steps of the gradient expansion

$$\Big\langle\mathcal{A}_{DET}[\hat{T},\hat{\mathcal{V}};\hat{\mathfrak{J}}]\Big\rangle_{\hat{\mathcal{V}}}=\frac{1}{2}\Big\langle\underset{\int_C d^4x_p\ \eta_p}{\overset{a(=1,2)}{\mathrm{TR}}}\ \underset{N_f,\hat{\gamma}_{mn}^{(\mu)},N_c}{\mathfrak{tr}}\Big(\ln\overbrace{\Big[\hat{\mathcal{H}}+\Delta\hat{\mathcal{H}}+\widetilde{\mathfrak{J}}(\hat{T}^{-1},\hat{T})\Big]}^{\hat{\mathcal{O}}_{N;M}^{ba}(y_q,x_p)}-\ln\big[\hat{\mathcal{H}}\big]\Big)\Big\rangle_{\hat{\mathcal{V}}} \tag{D.29}$$

$$=\frac{1}{2}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\Big\langle\underset{\int_C d^4x_p\ \eta_p}{\overset{a(=1,2)}{\mathrm{TR}}}\ \underset{N_f,\hat{\gamma}_{mn}^{(\mu)},N_c}{\mathfrak{tr}}\Big[\Big[\Big(\Delta\hat{\mathcal{H}}+\widetilde{\mathfrak{J}}(\hat{T}^{-1},\hat{T})\Big)\big(\hat{\mathcal{H}}^{-1}\big)\Big]^{n}\Big]\Big\rangle_{\hat{\mathcal{V}}}$$

$$=\frac{1}{2}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\Big\langle\underset{\int_C d^4x_p\ \eta_p}{\overset{a(=1,2)}{\mathrm{TR}}}\ \underset{N_f,\hat{\gamma}_{mn}^{(\mu)},N_c}{\mathfrak{tr}}\Big[\Big[\Delta\hat{\mathcal{H}}\ \hat{\mathcal{H}}^{-1}\Big]^{n}\Big]\Big\rangle_{\hat{\mathcal{V}}}+$$

$$+\ \Big\langle\underset{\int_C d^4x_p\ \eta_p}{\overset{a(=1,2)}{\mathrm{TR}}}\ \underset{N_f,\hat{\gamma}_{mn}^{(\mu)},N_c}{\mathfrak{tr}}\Big[\widetilde{\mathfrak{J}}(\hat{T}^{-1},\hat{T})\Big)\ \hat{\mathcal{H}}^{-1}\ \Big(\frac{1}{2}\sum_{n=1}^{\infty}(-1)^{n+1}\Big[\Delta\hat{\mathcal{H}}\ \hat{\mathcal{H}}^{-1}\Big]^{n-1}\Big)\Big]\Big\rangle_{\hat{\mathcal{V}}}+\Big\langle\mathsf{O}\Big[\big(\widetilde{\mathfrak{J}}(\hat{T}^{-1},\hat{T})\big)^{2}\Big]\Big\rangle_{\hat{\mathcal{V}}};$$

$$\widetilde{\mathfrak{J}}_{N;M}^{ba}\big(\hat{T}^{-1}(y_q)\,,\,\hat{T}(x_p)\big)\ =\ \Big(\hat{T}^{-1}(y_q)\ \hat{I}\ \hat{S}\ \eta_q\ \frac{\hat{\mathfrak{J}}_{N';M'}^{b'a'}(y_q,x_p)}{\mathcal{N}}\ \eta_p\ \hat{S}\ \hat{I}\ \hat{T}(x_p)\Big)_{N;M}^{ba}\,. \tag{D.30}$$

Apart from the propagation rules (D.25-D.28), one has to include the proper action of unsaturated gradients $\hat{\boldsymbol{\partial}}_{\boldsymbol{p},\boldsymbol{\mu}}$ onto the background potential $\mathcal{V}_{\alpha;\kappa}(x_p)$ (3.60,D.8), originating from the inverse operators $\hat{\mathcal{H}}^{-1}$ or their corresponding background field averaged Green functions $\hat{G}^{(0)}$ (D.10-D.13). One has to take into account the specific commutator relation $[\hat{\boldsymbol{\partial}}_{\boldsymbol{p},\boldsymbol{\kappa}}\,,\hat{\mathcal{H}}^{-1}]_-$ between the unsaturated derivative $\hat{\boldsymbol{\partial}}_{\boldsymbol{p},\boldsymbol{\kappa}}$ and the anomalous doubled inverse operator $\hat{\mathcal{H}}^{-1}$ (D.10-D.13) in order to incorporate a proper averaging with path integral (3.59) over derivatives of background potentials as $\langle\,(\hat{\partial}_{p,\kappa}\mathcal{V}_{\alpha;\kappa_1}(x_p)\,)\,\rangle_{\hat{\mathcal{V}}}$ (or their higher order products as e.g. $\langle\,(\hat{\partial}_{p,\kappa}\mathcal{V}_{\alpha;\kappa_1}(x_p)\,)\;(\hat{\partial}_{p,\lambda}\mathcal{V}_{\beta;\lambda_1}(x_p)\,)\,\rangle_{\hat{\mathcal{V}}}$). We have therefore to conclude from Eqs. (D.31) the final commutator $[\hat{\boldsymbol{\partial}}_{\boldsymbol{p},\boldsymbol{\kappa}}\,,\hat{\mathcal{H}}^{-1}]_-$ which yields an additional saturated derivative of the anomalous doubled background potential $\hat{\mathfrak{V}}(x_p)=\mathcal{V}_{\alpha;\kappa_1}(x_p)\;\hat{V}_\alpha^{\kappa_1}$ (D.8) propagating with $\hat{\mathcal{H}}^{-1}$ or approximately with products of $\hat{G}^{(0)}=\langle\hat{\mathcal{H}}^{-1}\rangle_{\hat{\mathcal{V}}}$

$$\begin{aligned}
\hat{\boldsymbol{\partial}}_{\boldsymbol{p},\boldsymbol{\kappa}}\,\hat{\mathcal{H}}\,\hat{\mathcal{H}}^{-1} &= \hat{\boldsymbol{\partial}}_{\boldsymbol{p},\boldsymbol{\kappa}}\;\hat{1}\;; & \hat{\boldsymbol{\partial}}_{\boldsymbol{p},\boldsymbol{\kappa}} &= \hat{\mathcal{H}}\;\hat{\boldsymbol{\partial}}_{\boldsymbol{p},\boldsymbol{\kappa}}\;\hat{\mathcal{H}}^{-1}+\Big[\hat{\boldsymbol{\partial}}_{\boldsymbol{p},\boldsymbol{\kappa}}\,,\hat{\mathcal{H}}\Big]_-\;\hat{\mathcal{H}}^{-1}\;; \\
\Big[\hat{\boldsymbol{\partial}}_{\boldsymbol{p},\boldsymbol{\kappa}}\,,\hat{\mathcal{H}}^{-1}\Big]_- &= -\hat{\mathcal{H}}^{-1}\;\Big[\hat{\boldsymbol{\partial}}_{\boldsymbol{p},\boldsymbol{\kappa}}\,,\hat{\mathcal{H}}\Big]_-\;\hat{\mathcal{H}}^{-1}\;; & \Big[\hat{\boldsymbol{\partial}}_{\boldsymbol{p},\boldsymbol{\kappa}}\,,\hat{\mathcal{H}}\Big]_- &= \Big(\hat{\partial}_{p,\kappa}\hat{\mathfrak{V}}(x_p)\Big)=\big(\hat{\partial}_{p,\kappa}\mathcal{V}_{\alpha;\kappa_1}\big)\;\hat{V}_\alpha^{\kappa_1}\;; \\
\Big[\hat{\boldsymbol{\partial}}_{\boldsymbol{p},\boldsymbol{\kappa}}\,,\hat{\mathcal{H}}^{-1}\Big]_- &= -\,\hat{\mathcal{H}}^{-1}\;\underbrace{\big(\hat{\partial}_{p,\boldsymbol{\kappa}}\mathcal{V}_{\alpha;\kappa_1}\big)\;\hat{V}_\alpha^{\kappa_1}}_{(\,\hat{\partial}_{p,\boldsymbol{\kappa}}\hat{\mathfrak{V}}\,)}\;\hat{\mathcal{H}}^{-1}\;; & \Big[\hat{\boldsymbol{\partial}}_{\boldsymbol{p},\boldsymbol{\kappa}}\,,\hat{G}^{(0)}\Big]_- &\simeq -\,\hat{G}^{(0)}\;\underbrace{\big(\hat{\partial}_{p,\boldsymbol{\kappa}}\mathcal{V}_{\alpha;\kappa_1}\big)\;\hat{V}_\alpha^{\kappa_1}}_{(\,\hat{\partial}_{p,\boldsymbol{\kappa}}\hat{\mathfrak{V}}\,)}\;\hat{G}^{(0)}\;;
\end{aligned} \tag{D.31}$$

$$\hat{V}_\alpha^{\kappa_1}=\begin{pmatrix}\big[\hat{\beta}(\imath\;\hat{\gamma}^{\kappa_1}\;\hat{t}_\alpha+\hat{m})\big] & 0 \\ 0 & \big[\hat{\beta}(\imath\;\hat{\gamma}^{\kappa_1}\;\hat{t}_\alpha+\hat{m})\big]^T\end{pmatrix}.$$

As we use the commutator (D.31) for the expansion (D.29), one proceeds to the correct expression (D.32) of the gradient expansion which also consists of derivatives of the background potentials following from the inverse operators $\hat{\mathcal{H}}^{-1}$ or their corresponding averaged Green functions (D.10-D.13). One has even to calculate commutators of $\hat{\mathcal{H}}^{-1}$ with multiple factors of unsaturated gradient operators in order to accomplish the correct transport functions of background potentials with their derivatives. However, these commutators of $\hat{\mathcal{H}}^{-1}$ with higher order gradient operators straightforwardly result from subsequent application of commutators (D.31). We list relation (D.32), derived from Eq. (D.29), as an intermediate step under single application of the commutator (D.31); nevertheless, further commutator extensions have to be computed for various factors of gradient operators (D.32) in order to achieve the complete set of transport functions with background potentials and their derivatives

$$\Big\langle\mathcal{A}_{DET}\big[\hat{T},\hat{\mathcal{V}};\hat{\mathcal{J}}\equiv 0\big]\Big\rangle_{\hat{\boldsymbol{\mathcal{V}}}}=\frac{1}{2}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\Big\langle\underset{\int_C d^4x_p\;\eta_p}{\overset{a(=1,2)}{\mathrm{TR}}}\;\underset{N_f,\hat{\gamma}_{mn}^{(\mu)},N_c}{\mathfrak{tr}}\Big[\Big(\Delta\hat{\mathcal{H}}\;\hat{\mathcal{H}}^{-1}\Big)^n\Big]\Big\rangle_{\hat{\boldsymbol{\mathcal{V}}}} \tag{D.32}$$

$$\begin{aligned}
&= \frac{1}{2}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\Big\langle\underset{\int_C d^4x_p\;\eta_p}{\overset{a(=1,2)}{\mathrm{TR}}}\;\underset{N_f,\hat{\gamma}_{mn}^{(\mu)},N_c}{\mathfrak{tr}}\Big[\Big[\Big(\delta\hat{\mathfrak{h}}+\delta\hat{\mathfrak{K}}^{\boldsymbol{\kappa}}\;\hat{\boldsymbol{\partial}}_{\boldsymbol{p},\boldsymbol{\kappa}}\Big)\;\hat{\mathcal{H}}^{-1}\Big]^n\Big]\Big\rangle_{\hat{\boldsymbol{\mathcal{V}}}} \\
&= \frac{1}{2}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\Big\langle\underset{\int_C d^4x_p\;\eta_p}{\overset{a(=1,2)}{\mathrm{TR}}}\;\underset{N_f,\hat{\gamma}_{mn}^{(\mu)},N_c}{\mathfrak{tr}}\Big[\Big[\delta\hat{\mathfrak{h}}\;\hat{\mathcal{H}}^{-1}+\delta\hat{\mathfrak{K}}^{\boldsymbol{\kappa}}\;\Big[\hat{\boldsymbol{\partial}}_{\boldsymbol{p},\boldsymbol{\kappa}}\,,\hat{\mathcal{H}}^{-1}\Big]_-+\delta\hat{\mathfrak{K}}^{\boldsymbol{\kappa}}\;\hat{\mathcal{H}}^{-1}\;\hat{\boldsymbol{\partial}}_{\boldsymbol{p},\boldsymbol{\kappa}}\Big]^n\Big]\Big\rangle_{\hat{\boldsymbol{\mathcal{V}}}} \\
&= \frac{1}{2}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\Big\langle\underset{\int_C d^4x_p\;\eta_p}{\overset{a(=1,2)}{\mathrm{TR}}}\;\underset{N_f,\hat{\gamma}_{mn}^{(\mu)},N_c}{\mathfrak{tr}}\Big[\Big[\Big(\delta\hat{\mathfrak{h}}-\delta\hat{\mathfrak{K}}^{\boldsymbol{\kappa}}\;\hat{\mathcal{H}}^{-1}\;\big(\hat{\partial}_{p,\boldsymbol{\kappa}}\mathcal{V}_{\alpha;\kappa_1}\big)\;\hat{V}_\alpha^{\kappa_1}\Big)\;\hat{\mathcal{H}}^{-1}+\delta\hat{\mathfrak{K}}^{\boldsymbol{\kappa}}\;\hat{\mathcal{H}}^{-1}\;\hat{\boldsymbol{\partial}}_{\boldsymbol{p},\boldsymbol{\kappa}}\Big]^n\Big]\Big\rangle_{\hat{\boldsymbol{\mathcal{V}}}} \\
&= \frac{1}{2}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\Big\langle\underset{\int_C d^4x_p\;\eta_p}{\overset{a(=1,2)}{\mathrm{TR}}}\;\underset{N_f,\hat{\gamma}_{mn}^{(\mu)},N_c}{\mathfrak{tr}}\Big[\Big(\Big[\hat{T}^{-1}\;\hat{S}\;\big(\hat{\beta}\;\hat{\partial\!\!\!/}_p\hat{T}\big)+\mathcal{V}_{\alpha;\kappa_1}\;\big(\hat{T}^{-1}\;\hat{V}_\alpha^{\kappa_1}\;\hat{T}-\hat{V}_\alpha^{\kappa_1}\big)+ \\
&\quad -\;\delta\hat{\mathfrak{K}}^{\boldsymbol{\kappa}}\;\hat{\mathcal{H}}^{-1}\;\big(\hat{\partial}_{p,\boldsymbol{\kappa}}\mathcal{V}_{\alpha;\kappa_1}\big)\;\hat{V}_\alpha^{\kappa_1}\Big]\;\hat{\mathcal{H}}^{-1}+\delta\hat{\mathfrak{K}}^{\boldsymbol{\kappa}}\;\hat{\mathcal{H}}^{-1}\;\hat{\boldsymbol{\partial}}_{\boldsymbol{p},\boldsymbol{\kappa}}\Big)^n\Big]\Big\rangle_{\hat{\boldsymbol{\mathcal{V}}}}
\end{aligned}$$

$$= \frac{1}{2}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\Big\langle \underset{\int_C d^4x_p\;\eta_p}{\overset{a(=1,2)}{\mathrm{TR}}}\;\underset{N_f,\hat{\gamma}_{mn}^{(\mu)},N_c}{\mathfrak{tr}}\Big[\Big[\boldsymbol{\Delta'}\hat{\mathfrak{h}}\;\hat{\mathcal{H}}^{-1}+\delta\hat{\mathcal{K}}^{\boldsymbol{\kappa}}\;\hat{\mathcal{H}}^{-1}\;\hat{\boldsymbol{\partial}}_{\boldsymbol{p},\boldsymbol{\kappa}}\Big]^n\Big]\Big\rangle_{\hat{\boldsymbol{\mathcal{V}}}}\,.$$

In order to simplify expressions, we introduce the abbreviations (D.33-D.38) for various parts of the gradient operators $\Delta\hat{\mathcal{H}}$ and $\Delta\hat{\mathcal{H}}\;\hat{\mathcal{H}}^{-1}\simeq\Delta\hat{\mathcal{H}}\;\hat{G}^{(0)}$, which occur in the expansion of the logarithm in (D.32), and define the symbol $\boldsymbol{\Delta'}\hat{\mathfrak{h}}$ (D.33,D.38) for a part of $\Delta\hat{\mathcal{H}}\;\hat{\mathcal{H}}^{-1}\simeq\Delta\hat{\mathcal{H}}\;\hat{G}^{(0)}$ in the last line of (D.32). The equations (D.6-D.9) determine the saturated gradient part $\hat{\mathfrak{k}}(x_p)=\hat{T}^{-1}(x_p)\,\hat{S}\,(\beta\,\hat{\not\partial}_p\hat{T}(x_p)\,)$ and potential matrix part $\delta\hat{\mathfrak{V}}(x_p)=\hat{T}^{-1}(x_p)\hat{\mathfrak{V}}(x_p)\hat{T}(x_p)-\hat{\mathfrak{V}}(x_p)$, which are combined to $\delta\hat{\mathfrak{h}}(x_p)=\hat{\mathfrak{k}}(x_p)+\delta\hat{\mathfrak{V}}(x_p)$ (D.7), in addition we discern the 'unsaturated' gradient operator part $\delta\hat{\mathcal{K}}^{\boldsymbol{\kappa}}(x_p)\;\hat{\boldsymbol{\partial}}_{\boldsymbol{p},\boldsymbol{\kappa}}$ (D.9). Moreover, the commutator $[\hat{\boldsymbol{\partial}}_{\boldsymbol{p},\boldsymbol{\kappa}}\;\boldsymbol{,}\;\hat{\mathcal{H}}^{-1}]_-$ (D.31) in (D.32) gives rise to the potential matrix term $(\hat{\partial}'\hat{\mathcal{V}})\;\hat{\mathcal{H}}^{-1}=(\delta\hat{\mathcal{K}}^{\boldsymbol{\kappa}}\;\hat{\mathcal{H}}^{-1}\;(\hat{\partial}_{p,\boldsymbol{\kappa}}\mathcal{V}_{\alpha;\kappa_1})\;\hat{V}_\alpha^{\kappa_1})\;\hat{\mathcal{H}}^{-1}$ (D.33) which we add to $\delta\hat{\mathfrak{h}}(x_p)=\hat{\mathfrak{k}}(x_p)+\delta\hat{\mathfrak{V}}(x_p)$ (D.7) for defining a new saturated gradient operator part $\boldsymbol{\Delta'}\hat{\mathfrak{h}}(x_p)$ (D.33,D.34). Nevertheless, we have to distinguish between the presence or abscence of the anomalous doubled inverse operator $\hat{\mathcal{H}}^{-1}$ or averaged propagator $\hat{G}^{(0)}$ so that one has two potential matrix terms $\hat{\partial}'\hat{\mathcal{V}}$ and $\hat{\partial}\hat{\mathcal{V}}(x_p)$ from the commutators (D.31) where the missing prime " $\boldsymbol{'}$ " of the latter indicates the missing of the inverse operator $\hat{\mathcal{H}}^{-1}$ or averaged propagator $\hat{G}^{(0)}$ (compare relations (D.33) and (D.34) !). This notation with the supplementary prime is also transferred to the total saturated gradient parts $\boldsymbol{\Delta'}\hat{\mathfrak{h}}\;\hat{\mathcal{H}}^{-1}$, $\Delta\hat{\mathfrak{h}}\;\hat{\mathcal{H}}^{-1}$ with the corresponding different potential matrix terms $\boldsymbol{\Delta'}\hat{\mathfrak{V}}(x_p)$ (D.35), $\Delta\hat{\mathfrak{V}}(x_p)$ (D.36). The potential matrix term $\boldsymbol{\Delta'}\hat{\mathfrak{V}}(x_p)$ consists of the sum of $\delta\hat{\mathfrak{V}}(x_p)$ from $\Delta\hat{\mathcal{H}}(x_p)$ and $-\hat{\partial}'\hat{\mathcal{V}}$, originating from the commutator $[\hat{\boldsymbol{\partial}}_{\boldsymbol{p},\boldsymbol{\kappa}}\boldsymbol{,}\hat{\mathcal{H}}^{-1}]_-=-(\,\hat{\partial}'\hat{\mathcal{V}}\,)\;\hat{\mathcal{H}}^{-1}=-(\,\delta\hat{\mathcal{K}}^{\boldsymbol{\kappa}}\;\hat{\mathcal{H}}^{-1}\;(\hat{\partial}_{p,\boldsymbol{\kappa}}\mathcal{V}_{\alpha;\kappa_1})\,\hat{V}_\alpha^{\kappa_1}\,)\;\hat{\mathcal{H}}^{-1}$, whereas $\Delta\hat{\mathfrak{V}}(x_p)$ is defined as the sum of $\delta\hat{\mathfrak{V}}(x_p)$ from $\Delta\hat{\mathcal{H}}(x_p)$ and $-(\hat{\partial}\hat{\mathcal{V}}(x_p)\,)=-(\hat{\partial}_{p,\boldsymbol{\kappa}}\mathcal{V}_{\alpha;\kappa_1}(x_p)\,)\,\delta\hat{\mathcal{K}}^{\boldsymbol{\kappa}}(x_p)\;\hat{V}_\alpha^{\kappa_1}$ without any propagator terms $\hat{\mathcal{H}}^{-1}$ or $\hat{G}^{(0)}$. Therefore, we have abbreviated the total gradient operator $\Delta\hat{\mathcal{H}}\;\hat{\mathcal{H}}^{-1}$ of the logarithm in (D.32) by $\boldsymbol{\Delta'}\hat{\mathfrak{h}}\;\hat{\mathcal{H}}^{-1}+\delta\hat{\mathcal{K}}^{\boldsymbol{\kappa}}\;\hat{\mathcal{H}}^{-1}\;\hat{\boldsymbol{\partial}}_{\boldsymbol{p},\boldsymbol{\kappa}}$ (D.33,D.38) with a prime, but have to apply the unprimed versions $\Delta\hat{\mathfrak{h}}$ (D.34) and $\Delta\hat{\mathfrak{V}}(x_p)$ (D.36) in later steps of transformations where the operators $\hat{G}^{(0)}$, $\hat{\mathcal{H}}^{-1}$ are removed according to the assumed rules (D.25-D.28) for propagation of anomalous doubled, generalized fields as plane wave states or source fields $J_{\psi;M}^a(x_p)$

$$\begin{aligned}
\boldsymbol{\Delta'}\hat{\mathfrak{h}} &= \hat{\mathfrak{k}}+\boldsymbol{\Delta'}\hat{\mathfrak{V}}=\hat{\mathfrak{k}}+\delta\hat{\mathfrak{V}}-\hat{\partial}'\hat{\mathcal{V}}\;; && \text{(D.33)}\\
\hat{\partial}'\hat{\mathcal{V}} &= \delta\hat{\mathcal{K}}^{\boldsymbol{\kappa}}\;\hat{\mathcal{H}}^{-1}\;\big(\hat{\partial}_{p,\boldsymbol{\kappa}}\mathcal{V}_{\alpha;\kappa_1}\big)\;\hat{V}_\alpha^{\kappa_1}=\delta\hat{\mathcal{K}}^{\boldsymbol{\kappa}}\;\hat{\mathcal{H}}^{-1}\;\big(\hat{\partial}_{p,\boldsymbol{\kappa}}\hat{\mathfrak{V}}\big)\;; &&\\
\Delta\hat{\mathfrak{h}}(x_p) &= \hat{\mathfrak{k}}(x_p)+\Delta\hat{\mathfrak{V}}(x_p)=\hat{\mathfrak{k}}(x_p)+\delta\hat{\mathfrak{V}}(x_p)-(\hat{\partial}\hat{\mathcal{V}}(x_p)\,)\;; && \text{(D.34)}\\
(\hat{\partial}\hat{\mathcal{V}}(x_p)\,) &= \big(\hat{\partial}_{p,\boldsymbol{\kappa}}\mathcal{V}_{\alpha;\kappa_1}(x_p)\big)\;\delta\hat{\mathcal{K}}^{\boldsymbol{\kappa}}(x_p)\;\hat{V}_\alpha^{\kappa_1}=\delta\hat{\mathcal{K}}^{\boldsymbol{\kappa}}(x_p)\;\big(\hat{\partial}_{p,\boldsymbol{\kappa}}\hat{\mathfrak{V}}(x_p)\big)\;; &&\\
\boldsymbol{\Delta'}\hat{\mathfrak{V}} &= \delta\hat{\mathfrak{V}}-\hat{\partial}'\hat{\mathcal{V}} && \text{(D.35)}\\
&= \big(\hat{T}^{-1}\;\hat{\mathfrak{V}}\;\hat{T}-\hat{\mathfrak{V}}\big)-\delta\hat{\mathcal{K}}^{\boldsymbol{\kappa}}\;\hat{\mathcal{H}}^{-1}\;\big(\hat{\partial}_{p,\boldsymbol{\kappa}}\mathcal{V}_{\alpha;\kappa_1}\big)\;\hat{V}_\alpha^{\kappa_1}\;; &&\\
\Delta\hat{\mathfrak{V}}(x_p) &= \delta\hat{\mathfrak{V}}(x_p)-(\hat{\partial}\hat{\mathcal{V}}(x_p)\,) && \text{(D.36)}\\
&= \big(\hat{T}^{-1}(x_p)\;\hat{\mathfrak{V}}(x_p)\;\hat{T}(x_p)-\hat{\mathfrak{V}}(x_p)\big)-\big(\hat{\partial}_{p,\boldsymbol{\kappa}}\mathcal{V}_{\alpha;\kappa_1}(x_p)\big)\;\delta\hat{\mathcal{K}}^{\boldsymbol{\kappa}}(x_p)\;\hat{V}_\alpha^{\kappa_1}\;; &&\\
\delta\hat{\mathfrak{h}}(x_p) &= \hat{\mathfrak{k}}(x_p)+\delta\hat{\mathfrak{V}}(x_p)\;; && \text{(D.37)}\\
\hat{\mathfrak{k}}(x_p) &= \hat{T}^{-1}(x_p)\;\hat{S}\;\beta\;\hat{\gamma}^\mu\big(\hat{\partial}_{p,\mu}\hat{T}(x_p)\big)\;; &&\\
\Delta\hat{\mathcal{H}}\;\hat{\mathcal{H}}^{-1} &= \boldsymbol{\Delta'}\hat{\mathfrak{h}}\;\hat{\mathcal{H}}^{-1}+\delta\hat{\mathcal{K}}^{\boldsymbol{\kappa}}\;\hat{\mathcal{H}}^{-1}\;\hat{\boldsymbol{\partial}}_{\boldsymbol{p},\boldsymbol{\kappa}}\,. && \text{(D.38)}
\end{aligned}$$

The background averaging with (3.59,D.2) in (D.32) is split into independent averages over single factors of the doubled Green function (D.10-D.24) with remaining background averaging over factors of $(\,\hat{\partial}_{p,\kappa}\mathcal{V}_{\alpha;\kappa_1}(x_p)\,)$ and over $\hat{\mathfrak{V}}(x_p)$ (D.8) in $\boldsymbol{\Delta'}\hat{\mathfrak{h}}$ (D.33). One therefore obtains the described propagation with the doubled non-equilibrium Green functions (D.10-D.24) back to the initial spacetime point. If we assume the appearance of changing or non-diagonal contour time indices $p\neq q$ for Green functions as ${}^{a=1}\widehat{\langle x_-}|\hat{G}^{(0)}|\widehat{y_+\rangle}^{a=1}\neq 0$ in the gradient expansion of $\langle\mathcal{A}_{DET}[\hat{T},\hat{\boldsymbol{\mathcal{V}}};\hat{\mathcal{J}}\equiv 0]\rangle_{\hat{\boldsymbol{\mathcal{V}}}}$, one has also to consider a back propagation, as e.g. with ${}^{a=1}\widehat{\langle y_+}|\hat{G}^{(0)}|\widehat{x_-\rangle}^{a=1}$, which vanishes completely due to the defined, contour extended, doubled Heaviside functions (D.20-D.24). In consequence one only has a propagation on a fixed branch 'p' of

the time contour in the gradient expansion (D.29-D.32) [12]

$$\text{product of time contour Green functions } \left({}^{a}\widehat{\langle x_p}|\hat{G}^{(0)}|\widehat{y_q\rangle}^{b}\right) \text{ in } \Big\langle\mathcal{A}_{DET}[\hat{T},\hat{\mathcal{V}};\hat{\mathcal{J}}\equiv 0]\Big\rangle_{\hat{\boldsymbol{\mathcal{V}}}} \to \text{remaining terms}$$
$$\to \quad \boldsymbol{\delta_{pq}} \text{ product of Green functions on fixed branch 'p' of time contour } \left(\delta_{ab}\;{}^{a}\widehat{\langle x_p}|\hat{G}^{(0)}|\widehat{y_p\rangle}^{a}\right).$$

According to the propagation back to the same spacetime point in $\langle\mathcal{A}_{DET}[\hat{T},\hat{\mathcal{V}};\hat{\mathcal{J}}\equiv 0]\rangle_{\hat{\mathcal{V}}}$, one has to regard at least two or any higher, even-numbered, non-diagonal factors of gradient terms $\Delta\hat{\mathfrak{H}}^{a\neq b}(x_p)$ (D.6-D.9), except for the first order case $n=1$ in (D.32). One has to distinct between the block diagonal or off-diagonal gradient terms (D.39,D.40) by the anti-commutator $\frac{1}{2}\hat{S}\,\{\Delta\hat{\mathfrak{H}}\,,\hat{S}\}_+$ or commutator $-\frac{1}{2}\hat{S}\,[\Delta\hat{\mathfrak{H}}\,,\hat{S}]_-$ parts with projection matrix $\hat{S}$ (2.21) of the coset space, respectively

$$\Delta\hat{\mathfrak{H}}^{a=b}(x_p) = \frac{1}{2}\Big(\hat{S}\;\Delta\hat{\mathfrak{H}}(x_p)\;\hat{S}+\Delta\hat{\mathfrak{H}}(x_p)\Big)=\frac{1}{2}\;\hat{S}\;\Big\{\Delta\hat{\mathfrak{H}}(x_p)\,,\hat{S}\Big\}_+; \tag{D.39}$$
$$\Delta\hat{\mathfrak{H}}^{a\neq b}(x_p) = -\frac{1}{2}\Big(\hat{S}\;\Delta\hat{\mathfrak{H}}(x_p)\;\hat{S}-\Delta\hat{\mathfrak{H}}(x_p)\Big)=-\frac{1}{2}\;\hat{S}\;\Big[\Delta\hat{\mathfrak{H}}(x_p)\,,\hat{S}\Big]_-. \tag{D.40}$$

This distinction is necessary because the propagation with only block diagonal operators $\Delta\hat{\mathfrak{H}}^{a=b}(x_p)=\frac{1}{2}\hat{S}\,\{\Delta\hat{\mathfrak{H}}\,,\hat{S}\}_+$ contributes vanishing terms in spacetime integrations over the Hilbert space trace and inserted unit operators according to back propagation to the initial, contour extended spacetime point. This particular, '*vanishing*' part is symbolically subtracted in (D.41) in order to point out the importance of other, remaining, non-vanishing combinations with $\frac{1}{2}\hat{S}\,\{\Delta\hat{\mathfrak{H}}\,,\hat{S}\}_+$ and $-\frac{1}{2}\hat{S}\,[\Delta\hat{\mathfrak{H}}\,,\hat{S}]_-$

$$\begin{aligned}
\Big\langle\mathcal{A}_{DET}[\hat{T},\hat{\mathcal{V}};\hat{\mathcal{J}}\equiv 0]\Big\rangle_{\hat{\boldsymbol{\mathcal{V}}}} &= \frac{1}{2}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\Big\langle\underset{\int_C d^4x_p\;\eta_p}{\overset{a(=1,2)}{\mathrm{TR}}}\;\underset{N_f,\hat{\gamma}_{mn}^{(\mu)},N_c}{\mathfrak{tr}}\Big[\Big(\Delta\hat{\mathfrak{H}}\;\hat{G}^{(0)}\Big)^n\Big]\Big\rangle_{\hat{\boldsymbol{\mathcal{V}}}}+ \\
&- \underbrace{\frac{1}{2}\sum_{n=2}^{\infty}\frac{(-1)^{n+1}}{n}\Big\langle\underset{\int_C d^4x_p\;\eta_p}{\overset{a(=1,2)}{\mathrm{TR}}}\;\underset{N_f,\hat{\gamma}_{mn}^{(\mu)},N_c}{\mathfrak{tr}}\Big[\Big(\frac{1}{2}\;\hat{S}\{\Delta\hat{\mathfrak{H}}\,,\hat{S}\}_+\;\hat{G}^{(0)}\Big)^n\Big]\Big\rangle_{\hat{\boldsymbol{\mathcal{V}}}}}_{\equiv 0\text{, vanishing contribution in spacetime integrations !}} \\
&= \frac{1}{2}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\Big\langle\underset{\int_C d^4x_p\;\eta_p}{\overset{a(=1,2)}{\mathrm{TR}}}\;\underset{N_f,\hat{\gamma}_{mn}^{(\mu)},N_c}{\mathfrak{tr}}\Big[\Big(\hat{T}^{-1}\;\hat{S}\;(\hat{\beta}\;\hat{\mathscr{\partial}}_p\hat{T})\;\hat{G}^{(0)}+\mathcal{V}_{\alpha;\kappa_1}\;\big(\hat{T}^{-1}\;\hat{V}_\alpha^{\kappa_1}\;\hat{T}-\hat{V}_\alpha^{\kappa_1}\big)\;\hat{G}^{(0)}+ \\
&- \delta\hat{\mathfrak{K}}^{\boldsymbol{\kappa}}\;\hat{G}^{(0)}\;\big(\hat{\partial}_{p,\boldsymbol{\kappa}}\mathcal{V}_{\alpha;\kappa_1}\big)\;\hat{V}_\alpha^{\kappa_1}\;\hat{G}^{(0)}+\delta\hat{\mathfrak{K}}^{\boldsymbol{\kappa}}\;\hat{G}^{(0)}\;\hat{\boldsymbol{\partial}}_{\boldsymbol{p},\boldsymbol{\kappa}}\Big)^n\Big]\Big\rangle_{\hat{\boldsymbol{\mathcal{V}}}}+ \\
&- \frac{1}{2}\sum_{n=2}^{\infty}\frac{(-1)^{n+1}}{n}\Big\langle\underset{\int_C d^4x_p\;\eta_p}{\overset{a(=1,2)}{\mathrm{TR}}}\;\underset{N_f,\hat{\gamma}_{mn}^{(\mu)},N_c}{\mathfrak{tr}}\Big[\Big(\frac{\hat{S}}{2}\Big\{\hat{T}^{-1}\;\hat{S}\;(\hat{\beta}\;\hat{\mathscr{\partial}}_p\hat{T})+\mathcal{V}_{\alpha;\kappa_1}\;\big(\hat{T}^{-1}\;\hat{V}_\alpha^{\kappa_1}\;\hat{T}-\hat{V}_\alpha^{\kappa_1}\big)\,,\hat{S}\Big\}_+\;\hat{G}^{(0)}+ \\
&- \frac{\hat{S}}{2}\{\delta\hat{\mathfrak{K}}^{\boldsymbol{\kappa}}\,,\hat{S}\}_+\;\hat{G}^{(0)}\;\big(\hat{\partial}_{p,\boldsymbol{\kappa}}\mathcal{V}_{\alpha;\kappa_1}\big)\;\hat{V}_\alpha^{\kappa_1}\;\hat{G}^{(0)}+\frac{\hat{S}}{2}\{\,\delta\hat{\mathfrak{K}}^{\boldsymbol{\kappa}}\,,\hat{S}\}_+\;\hat{G}^{(0)}\;\hat{\boldsymbol{\partial}}_{\boldsymbol{p},\boldsymbol{\kappa}}\Big)^n\Big]\Big\rangle_{\hat{\boldsymbol{\mathcal{V}}}} \\
&= \frac{1}{2}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\Big\langle\underset{\int_C d^4x_p\;\eta_p}{\overset{a(=1,2)}{\mathrm{TR}}}\;\underset{N_f,\hat{\gamma}_{mn}^{(\mu)},N_c}{\mathfrak{tr}}\Big[\Big[\Big(\hat{\mathfrak{k}}+\delta\hat{\mathfrak{V}}-\hat{\partial}'\hat{\mathcal{V}}\Big)\;\hat{G}^{(0)}+\delta\hat{\mathfrak{K}}^{\boldsymbol{\kappa}}\;\hat{G}^{(0)}\;\hat{\boldsymbol{\partial}}_{\boldsymbol{p},\boldsymbol{\kappa}}\Big]^n\Big]\Big\rangle_{\hat{\boldsymbol{\mathcal{V}}}}+
\end{aligned} \tag{D.41}$$

[12] In the case of disordered systems, one has to take into account propagations with $\hat{G}^{(0)}$ of varying time contour branches because the corresponding gradient operator is non-diagonal in contour time indices $p\neq q$ whereas the gradient operator $\Delta\hat{\mathfrak{H}}(x_p)$ in this paper is diagonal in the contour time indices.

$$-\;\frac{1}{2}\sum_{n=2}^{\infty}\frac{(-1)^{n+1}}{n}\underbrace{\Big\langle \underset{\int_C d^4x_p\;\eta_p}{\overset{a(=1,2)}{\mathrm{TR}}}\;\underset{N_f,\hat{\gamma}_{mn}^{(\mu)},N_c}{\mathfrak{tr}}\bigg[\bigg(\frac{1}{2}\,\hat{S}\Big\{\Big(\hat{\mathfrak{t}}+\delta\hat{\mathfrak{V}}-\hat{\partial}'\hat{\mathcal{V}}\Big)\,\hat{G}^{(0)}+\delta\hat{\mathcal{K}}^{\boldsymbol{\kappa}}\,\hat{G}^{(0)}\,\hat{\boldsymbol{\partial}}_{\boldsymbol{p},\boldsymbol{\kappa}}\,,\,\hat{S}\Big\}_{+}\bigg)^{n}\bigg]\Big\rangle_{\hat{\mathcal{V}}}}_{\equiv 0,\text{ vanishing contribution in spacetime integrations !}}\;.$$

Corresponding to the definitions and notations of indices for the internal spaces in section 2, we group again the collective indices M_i, N_i which are composed of the isospin- (flavour-) indices f_i, g_i, the 4×4 Dirac gamma matrix indices m_i, n_i and the colour matrix indices r_i, s_i. Furthermore, the total numbers $\mathcal{N}$, $\mathcal{N}_k$ of spacetime and momentum-energy points have to scale the spacetime and momentum-energy integrations in the gradient expansion of $\langle\mathcal{A}_{DET}[\hat{T},\hat{\mathcal{V}};\hat{\mathcal{J}}\equiv 0]\rangle_{\hat{\mathcal{V}}}$; hence we can perform a large N-limit with the total number of points $\mathcal{N}\,\mathcal{N}_k=(\,N_L/(2\pi)\,)^4$ of the underlying grids, separated into N_L discrete points for each of the 3+1 dimensions

$$\begin{array}{rclrcl}
M_i &=& \{f_i\,,\,m_i\,,\,r_i\} & N_i &=& \{g_i\,,\,n_i\,,\,s_i\} \\
f_i &=& \text{u(p), d(own), (s(trange))} & g_i &=& \text{u(p), d(own), (s(trange))} \\
f_i &\overset{\text{or}}{=}& 1,\ldots,N_f & g_i &\overset{\text{or}}{=}& 1,\ldots,N_f \\
m_i &=& 1,\ldots,4\quad;[\hat{\gamma}^{\mu}]_{m_i,n_j} & n_i &=& 1,\ldots,4\quad;[\hat{\gamma}^{\nu}]_{m_j,n_i} \\
r_i &=& 1,\ldots,N_c=3 & s_i &=& 1,\ldots,N_c=3 \\
\mathcal{N} &=& \delta^{(4)}(0)=\frac{1}{\left(\Delta x\right)^4}=\frac{1}{\left(L/N_L\right)^4} & \mathcal{N}_k &=& \frac{1}{\left(\Delta k\right)^4}=\frac{1}{\left(2\pi/L\right)^4} \\
\mathcal{N}\,\mathcal{N}_k &=& \Big(N_L/(2\pi)\Big)^4 & \sum_{a_i=1,2} && \ldots\,.
\end{array}\tag{D.42}$$

The given notations (D.42) are used to label the internal spaces of the trace operations in $\langle\mathcal{A}_{DET}[\hat{T},\hat{\mathcal{V}};\hat{\mathcal{J}}\equiv 0]\rangle_{\hat{\mathcal{V}}}$ (D.43). According to the definitions of the anomalous doubled Hilbert states of spacetime, the complete matrix representation is specified for the gradient expansion of the determinant as a sum over the various orders n following from the logarithm. We emphasize the last line in (D.43) with Kronecker deltas over the internal symmetry spaces and the contour extended delta function of 3+1 dimensional spacetime which causes the propagation back to the initial spacetime and internal symmetry space state, considering the original traces from the determinant

$$\begin{aligned}
&\Big\langle\mathcal{A}_{DET}\big[\hat{T},\hat{\mathcal{V}};\hat{\mathcal{J}}\equiv 0\big]\Big\rangle_{\hat{\mathcal{V}}}=\frac{1}{2}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\Big\langle \underset{\int_C d^4x_p\;\eta_p}{\overset{a(=1,2)}{\mathrm{TR}}}\;\underset{N_f,\hat{\gamma}_{mn}^{(\mu)},N_c}{\mathfrak{tr}}\Big[\Big[\Delta\hat{\mathcal{H}}\;\hat{G}^{(0)}\Big]^{n}\Big]\Big\rangle_{\hat{\mathcal{V}}}= \\
&=\;\frac{1}{2}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\int_C d^4x_{p_n}^{(n)}\;\eta_{p_n}\;\mathcal{N}\sum_{a_n=1,2}\sum_{M_n}\prod_{i=0}^{n-1}\bigg(\int_C d^4x_{p_i}^{(i)}\;\eta_{p_i}\;\mathcal{N}\int_C d^4y_{q_i}^{(i)}\;\eta_{q_i}\;\mathcal{N}\sum_{a_i=1,2}\sum_{M_i}\sum_{N_i}\bigg)\times \\
&\times\;\bigg\langle\;{}^{a_n}\widehat{\langle x_{p_n}^{(n)}}|\Delta\hat{\mathcal{H}}_{M_n;N_{n-1}}|\widehat{y_{q_{n-1}}^{(n-1)}\rangle}^{a_{n-1}}\times{}^{a_{n-1}}\widehat{\langle y_{q_{n-1}}^{(n-1)}}|\hat{G}^{(0)}_{N_{n-1};M_{n-1}}|\widehat{x_{p_{n-1}}^{(n-1)}\rangle}^{a_{n-1}}\;\times \\
&\times\;{}^{a_{n-1}}\widehat{\langle x_{p_{n-1}}^{(n-1)}}|\Delta\hat{\mathcal{H}}_{M_{n-1};N_{n-2}}|\widehat{y_{q_{n-2}}^{(n-2)}\rangle}^{a_{n-2}}\times{}^{a_{n-2}}\widehat{\langle y_{q_{n-2}}^{(n-2)}}|\hat{G}^{(0)}_{N_{n-2};M_{n-2}}|\widehat{x_{p_{n-2}}^{(n-2)}\rangle}^{a_{n-2}}\;\times\;\ldots \\
&\times\;\ldots\;{}^{a_2}\widehat{\langle x_{p_2}^{(2)}}|\Delta\hat{\mathcal{H}}_{M_2;N_1}|\widehat{y_{q_1}^{(1)}\rangle}^{a_1}\times{}^{a_1}\widehat{\langle y_{q_1}^{(1)}}|\hat{G}^{(0)}_{N_1;M_1}|\widehat{x_{p_1}^{(1)}\rangle}^{a_1}\;\times \\
&\times\;{}^{a_1}\widehat{\langle x_{p_1}^{(1)}}|\Delta\hat{\mathcal{H}}_{M_1;N_0}|\widehat{y_{q_0}^{(0)}\rangle}^{a_0}\times{}^{a_0}\widehat{\langle y_{q_0}^{(0)}}|\hat{G}^{(0)}_{N_0;M_0}|\widehat{x_{p_0}^{(0)}\rangle}^{a_0}\bigg\rangle_{\hat{\mathcal{V}}}\;\times \\
&\times\;\delta_{p_np_0}\;\;\delta^{(4)}\big(x_{p_n}^{(n)}-x_{p_0}^{(0)}\big)\;\;\delta_{a_na_0}\;\;\delta_{f_nf_0}\;\;\delta_{m_nm_0}\;\;\delta_{r_nr_0}\;\;; \\
&\quad\Big(\text{summations in (D.43) without }\;a_n=a_{n-1}=\ldots=a_1=a_0!\Big)\;\;.
\end{aligned}\tag{D.43}$$

The trace operations of the action $\langle\mathcal{A}_{DET}[\hat{T},\hat{\mathcal{V}};\hat{\mathcal{J}}\equiv 0]\rangle_{\hat{\mathcal{V}}}$ also involve the Kronecker delta $\delta_{a_na_0}$ of the anomalous doubling. This particular Kronecker delta is substituted by the described anti-commutator (symbolized in Eq. (D.44)) with

projection matrix $\hat{S}$ of the coset space in a similar kind as for $\Delta\hat{\mathcal{H}}^{a=b}(x_p)$ (D.39)

$$\left(\begin{array}{c}(\text{field}')^* \\ (\text{field}')\end{array}\right)^{T,a_n} \underbrace{\frac{1}{2}\Big[\hat{S}\Big(\text{n-th order term}\Big)\hat{S}+\Big(\text{n-th order term}\Big)\Big]}_{\text{equivalent to } \delta_{a_n a_0}} \left(\begin{array}{c}(\text{field}) \\ (\text{field})^*\end{array}\right)^{a_0} = \tag{D.44}$$

$$= \left(\begin{array}{c}(\text{field}')^* \\ (\text{field}')\end{array}\right)^{T,a_n} \underbrace{\frac{\hat{S}}{2}\Big\{\Big(\text{n-th order term}\Big)\,,\,\hat{S}\Big\}_+}_{\text{equivalent to } \delta_{a_n a_0}} \left(\begin{array}{c}(\text{field}) \\ (\text{field})^*\end{array}\right)^{a_0} .$$

The gradient expansion of $\langle\mathcal{A}_{DET}[\hat{T},\hat{\mathcal{V}};\hat{\mathcal{J}}\equiv 0]\rangle_{\hat{\mathcal{V}}}$ is reduced to that of $\langle\mathcal{A}_{J_\psi}[\hat{T},\hat{\mathcal{V}};\hat{\mathcal{J}}\equiv 0]\rangle_{\hat{\mathcal{V}}}$ by factorization of the anomalous doubled, contour extended delta function of spacetime to plane wave states. Since we perform an expansion for lowest-order, gradually varying gradient terms, a cutoff momentum k_{max} is introduced in the anomalous doubled, contour extended momentum-energy integrations for the unit operator (compare appendix A)

$$\delta_{p_n p_0}\;\;\delta^{(4)}(x_{p_n}^{(n)}-x_{p_0}^{(0)})\;\;\delta_{a_n a_0} = \sum_{a=1,2}\int_C^{k_{max}} d^4k_p\;\eta_p\;\mathcal{N}_k\;\;{}^{a_0}\widehat{\langle x_{p_0}^{(0)}|\widehat{k_p}\rangle}^{a}\times{}^{a}\widehat{\langle\widehat{k_p}|x_{p_n}^{(n)}\rangle}^{a_n} = \tag{D.45}$$

$$= \;\;\delta_{a_n a_0}\int_C^{k_{max}} d^4k_p\;\eta_p\;\mathcal{N}_k\;\delta_{pp_0}\;\delta_{pp_n}\left\{\begin{array}{l}\exp\big\{\imath\,k_p\cdot(x_{p_0}^{(0)}-x_{p_n}^{(n)})\big\}\;;\;\text{for}\;\;;a_0=a_n=1\;; \\ \exp\big\{-\imath\,k_p\cdot(x_{p_0}^{(0)}-x_{p_n}^{(n)})\big\}\;;\;\text{for}\;\;;a_0=a_n=2\;.\end{array}\right.$$

One has to achieve a similar propagation as in $\langle\mathcal{A}_{J_\psi}[\hat{T},\hat{\mathcal{V}};\hat{\mathcal{J}}\equiv 0]\rangle_{\hat{\mathcal{V}}}$ due to relations (D.25-D.28) and therefore factorizes the Kronecker deltas of the internal spaces in the last line of the matrix representation for $\langle\mathcal{A}_{DET}[\hat{T},\hat{\mathcal{V}};\hat{\mathcal{J}}\equiv 0]\rangle_{\hat{\mathcal{V}}}$ (D.43)

$$\delta_{f_n f_0}=\sum_{\bar{f}=1}^{N_f}\delta_{f_n\bar{f}}\;\delta_{\bar{f}f_0}\;;\qquad \delta_{m_n m_0}=\sum_{\bar{m}=1}^{4}\delta_{m_n\bar{m}}\;\delta_{\bar{m}m_0}\;;\qquad \delta_{r_n r_0}=\sum_{\bar{r}=1}^{N_c}\delta_{r_n\bar{r}}\;\delta_{\bar{r}r_0}\;. \tag{D.46}$$

As we use the factorizations of the delta function and Kronecker deltas (D.45,D.46), one can define anomalous doubled plane wave states $\mathfrak{W}_{f_0,m_0,r_0}^{(\bar{f},\bar{m},\bar{r});a_0}(x_{p_0}^{(0)};k_p)$, $(\,\mathfrak{W}_{f_n,m_n,r_n}^{(\bar{f},\bar{m},\bar{r});a_0}(x_{p_n}^{(n)};k_p)\,)^\dagger$ in terms of (D.48,D.50) extended with their particular complex conjugates. The anomalous doubled plane wave states of four-momentum k_p replace the source fields $J_{\psi;N}^{\dagger b}(y_q)$, $J_{\psi;M}^{a}(x_p)$ and allow to apply rules (D.25-D.28) for propagation with Green function $\hat{G}^{(0)}$ averaged by the background functional (D.65,D.2). This is possible because the plane wave states and (D.46) decouple the trace operations leading to propagation back to the initial spacetime and internal space state

$$\mathfrak{W}_{f_0,m_0,r_0}^{(\bar{f},\bar{m},\bar{r});a_0}(x_{p_0}^{(0)};k_p) \;=\; \left(\begin{array}{c}\mathfrak{w}_{f_0,m_0,r_0}^{(\bar{f},\bar{m},\bar{r})}(x_{p_0}^{(0)};k_p) \\ \big(\mathfrak{w}_{f_0,m_0,r_0}^{(\bar{f},\bar{m},\bar{r})}(x_{p_0}^{(0)};k_p)\big)^*\end{array}\right)^{a_0}\;; \tag{D.47}$$

$$\mathfrak{w}_{f_0,m_0,r_0}^{(\bar{f},\bar{m},\bar{r})}(x_{p_0}^{(0)};k_p) \;=\; \exp\big\{\imath\,k_p\cdot x_{p_0}^{(0)}\big\}\;\delta_{pp_0}\;\delta_{\bar{f}f_0}\;\delta_{\bar{m}m_0}\;\delta_{\bar{r}r_0}\;; \tag{D.48}$$

$$\Big(\mathfrak{W}_{f_n,m_n,r_n}^{(\bar{f},\bar{m},\bar{r});a_0}(x_{p_n}^{(n)};k_p)\Big)^\dagger \;=\; \left(\begin{array}{c}\big(\mathfrak{w}_{f_n,m_n,r_n}^{(\bar{f},\bar{m},\bar{r})}(x_{p_n}^{(n)};k_p)\big)^* \\ \mathfrak{w}_{f_n,m_n,r_n}^{(\bar{f},\bar{m},\bar{r})}(x_{p_n}^{(n)};k_p)\end{array}\right)^{T,a_n}\;; \tag{D.49}$$

$$\mathfrak{w}_{f_n,m_n,r_n}^{(\bar{f},\bar{m},\bar{r})}(x_{p_n}^{(n)};k_p) \;=\; \exp\big\{\imath\,k_p\cdot x_{p_n}^{(n)}\big\}\;\delta_{pp_n}\;\delta_{\bar{f}f_n}\;\delta_{\bar{m}m_n}\;\delta_{\bar{r}r_n}\;. \tag{D.50}$$

We regroup the integrations of the matrix representation of $\langle\mathcal{A}_{DET}[\hat{T},\hat{\mathcal{V}};\hat{\mathcal{J}}\equiv 0]\rangle_{\hat{\mathcal{V}}}$ with inclusion of momentum-energy integrations and additional internal space summations so that one formally accomplishes an analogous action as $\langle\mathcal{A}_{J_\psi}[\hat{T},\hat{\mathcal{V}};\hat{\mathcal{J}}\equiv 0]\rangle_{\hat{\mathcal{V}}}$ where the source fields are substituted by the plane wave fields of (D.47-D.50)

$$\Big\langle\mathcal{A}_{DET}\big[\hat{T},\hat{\mathcal{V}};\hat{\mathcal{J}}\equiv 0\big]\Big\rangle_{\hat{\mathcal{V}}} = \int_C^{k_{max}} d^4k_p\;\eta_p\;\mathcal{N}_k\sum_{\bar{f}=1}^{N_f}\sum_{\bar{m}=1}^{4}\sum_{\bar{r}=1}^{N_c}\frac{1}{2}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\times \tag{D.51}$$

$$\times \int_C d^4x_{p_n}^{(n)}\;\eta_{p_n}\;\mathcal{N}\sum_{a_n=1,2}\sum_{M_n}\prod_{i=0}^{n-1}\bigg(\int_C d^4x_{p_i}^{(i)}\;\eta_{p_i}\;\mathcal{N}\int_C d^4y_{q_i}^{(i)}\;\eta_{q_i}\;\mathcal{N}\sum_{a_i=1,2}\sum_{M_i}\sum_{N_i}\bigg)\times$$
$$\times \bigg\langle\Big(\mathfrak{W}^{(\bar f,\bar m,\bar r);a_n}_{f_n,m_n,r_n}(x_{p_n}^{(n)};k_p)\Big)^\dagger\times{}^{a_n}\widehat{\langle x_{p_n}^{(n)}}|\Delta\hat{\mathcal{H}}_{M_n;N_{n-1}}|\widehat{y_{q_{n-1}}^{(n-1)}\rangle}^{a_{n-1}}\times{}^{a_{n-1}}\widehat{\langle y_{q_{n-1}}^{(n-1)}}|\hat G^{(0)}_{N_{n-1};M_{n-1}}|\widehat{x_{p_{n-1}}^{(n-1)}\rangle}^{a_{n-1}}\times$$
$$\times\;{}^{a_{n-1}}\widehat{\langle x_{p_{n-1}}^{(n-1)}}|\Delta\hat{\mathcal{H}}_{M_{n-1};N_{n-2}}|\widehat{y_{q_{n-2}}^{(n-2)}\rangle}^{a_{n-2}}\times{}^{a_{n-2}}\widehat{\langle y_{q_{n-2}}^{(n-2)}}|\hat G^{(0)}_{N_{n-2};M_{n-2}}|\widehat{x_{p_{n-2}}^{(n-2)}\rangle}^{a_{n-2}}\times\ldots$$
$$\times\;\ldots\;{}^{a_2}\widehat{\langle x_{p_2}^{(2)}}|\Delta\hat{\mathcal{H}}_{M_2;N_1}|\widehat{y_{q_1}^{(1)}\rangle}^{a_1}\times{}^{a_1}\widehat{\langle y_{q_1}^{(1)}}|\hat G^{(0)}_{N_1;M_1}|\widehat{x_{p_1}^{(1)}\rangle}^{a_1}\times$$
$$\times\;{}^{a_1}\widehat{\langle x_{p_1}^{(1)}}|\Delta\hat{\mathcal{H}}_{M_1;N_0}|\widehat{y_{q_0}^{(0)}\rangle}^{a_0}\times{}^{a_0}\widehat{\langle y_{q_0}^{(0)}}|\hat G^{(0)}_{N_0;M_0}|\widehat{x_{p_0}^{(0)}\rangle}^{a_0}\times\mathfrak{W}^{(\bar f,\bar m,\bar r);a_0}_{f_0,m_0,r_0}(x_{p_0}^{(0)};k_p)\bigg\rangle_{\hat{\mathcal{V}}}\times\;\delta_{a_na_0}\;\text{(compare Eq. (D.44))};$$
$$\Big(\text{summations in (D.51) without }\;a_n=a_{n-1}=\ldots=a_1=a_0\;!\Big)\;.$$

Direct application of rules (D.25-D.28) and commutators $[\hat{\boldsymbol{\partial}}_{\boldsymbol{p,\kappa}}\;,\;\hat G^{(0)}]_-$ (D.31) leads to relation (D.52) where the Green functions $\hat G^{(0)}$, averaged by background path integral (3.59,D.2), propagate the plane wave states $\mathfrak{W}^{(\bar f,\bar m,\bar r);a_0}_{f_0,m_0,r_0}(x_{p_0}^{(0)};k_p)$ on the right-hand side to the left, analogous 'source field' state $(\,\mathfrak{W}^{(\bar f,\bar m,\bar r);a_n}_{f_n,m_n,r_n}(x_{p_n}^{(n)};k_p)\,)^\dagger$ under subsequent action of the gradient operators $\Delta\hat{\mathcal{H}}^{a_{i+1}a_i}_{M_{i+1};M_i}(x_p)$. This process of propagation with $\hat G^{(0)}$ is complicated by the action of unsaturated gradients $\hat{\boldsymbol{\partial}}_{\boldsymbol{p,\kappa}}$ onto the background potential matrix $\hat{\mathfrak{V}}(x_p)$ (D.8) within the doubled Green functions. This requires intensive application of commutator relations (D.31) and results into additional transport coefficients with derivatives $(\hat\partial_{p,\kappa}\mathcal{V}_{\alpha;\kappa_1}(x_p)\,)$. The Kronecker delta $\delta_{a_na_0}$ for the anomalous doubling is taken into account by the anti-commutator $\frac{1}{2}\hat S\boldsymbol{\{}$ (n-th order term) $\boldsymbol{,}\,\hat S\boldsymbol{\}}_+$ (D.44) for every term 'n' in the expansion of the logarithm. Every factor with the gradient operator $\Delta\hat{\mathcal{H}}\;\hat G^{(0)}$ at n-th order $(\,\Delta\hat{\mathcal{H}}\;\hat G^{(0)}\,)^n$ is transformed by the commutator $[\hat{\boldsymbol{\partial}}_{\boldsymbol{p,\kappa}}\,\boldsymbol{,}\,\hat G^{(0)}]_-$ (D.31) so that one gains an extended relation (D.52) for $\langle\mathcal{A}_{DET}[\hat T,\hat{\mathcal{V}};\hat{\mathcal{J}}\equiv0]\rangle_{\hat{\mathcal{V}}}$; nevertheless, it is still necessary to transform unsaturated gradients $\hat{\boldsymbol{\partial}}_{\boldsymbol{p,\kappa}}$ and Green functions $\hat G^{(0)}$ in (D.52) by further application of extended commutator relations as $[\hat{\boldsymbol{\partial}}_{\boldsymbol{p,\kappa}}\;,\;\hat G^{(0)}]_-$ (D.31) in order to derive the complete, proper set of transport coefficients with the background potential $\mathcal{V}_{\alpha;\kappa_1}(x_p)$. After substitution of (D.33-D.38) into (D.51), we attain the action $\langle\mathcal{A}_{DET}[\hat T,\hat{\mathcal{V}};\hat{\mathcal{J}}\equiv0]\rangle_{\hat{\mathcal{V}}}$ with a single transformation of the commutator (D.31) as an intermediate step to be reduced by further applications of (D.31) and propagation rules (D.25-D.28)

$$\Big\langle\mathcal{A}_{DET}\big[\hat T,\hat{\mathcal{V}};\hat{\mathcal{J}}\equiv0\big]\Big\rangle_{\hat{\mathcal{V}}}=\int_C^{k_{max}}d^4k_p\;\eta_p\;\mathcal{N}_k\sum_{\bar f=1}^{N_f}\sum_{\bar m=1}^{4}\sum_{\bar r=1}^{N_c}\frac{1}{2}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\times \tag{D.52}$$
$$\times\int_C d^4x_{p_n}^{(n)}\;\mathcal{N}\sum_{a_n=1,2}\sum_{M_n}\prod_{i=0}^{n-1}\bigg(\int_C d^4x_{p_i}^{(i)}\;\mathcal{N}\sum_{a_i=1,2}\sum_{M_i}\sum_{N_i}\bigg)\bigg\langle\Big(\mathfrak{W}^{(\bar f,\bar m,\bar r);a_n}_{f_n,m_n,r_n}(x_{p_n}^{(n)};k_p)\Big)^\dagger\times$$
$$\times\;\frac{\hat S}{2}\bigg\{\bigg[\Big(\hat{\mathfrak{k}}(x_{p_n}^{(n)})+\delta\hat{\mathfrak{V}}(x_{p_n}^{(n)})\Big)^{a_na_{n-1}}_{M_n;N_{n-1}}\;{}^{a_{n-1}}\widehat{\langle x_{p_n}^{(n)}}|\hat G^{(0)}_{N_{n-1};M_{n-1}}|\widehat{x_{p_{n-1}}^{(n-1)}\rangle}^{a_{n-1}}+$$
$$-\;\delta\hat{\mathcal{K}}^{\boldsymbol{\kappa};a_na_{n-1}}_{M_n;N_{n-1}}(x_{p_n}^{(n)})\;\Big({}^{a_{n-1}}\widehat{\langle x_{p_n}^{(n)}}|\hat G^{(0)}\;\big(\hat\partial_{p,\boldsymbol{\kappa}}\hat{\mathfrak{V}}\big)\;\hat G^{(0)}|\widehat{x_{p_{n-1}}^{(n-1)}\rangle}^{a_{n-1}}\Big)_{N_{n-1};M_{n-1}}+$$
$$+\;\delta\hat{\mathcal{K}}^{\boldsymbol{\kappa};a_na_{n-1}}_{M_n;N_{n-1}}(x_{p_n}^{(n)})\;\;{}^{a_{n-1}}\widehat{\langle x_{p_n}^{(n)}}|\hat G^{(0)}_{N_{n-1};M_{n-1}}|\widehat{x_{p_{n-1}}^{(n-1)}\rangle}^{a_{n-1}}\;\hat{\boldsymbol{\partial}}^{\boldsymbol{(n-1)}}_{\boldsymbol{p_{n-1},\kappa}}\bigg]\times$$
$$\times\;\bigg[\Big(\hat{\mathfrak{k}}(x_{p_{n-1}}^{(n-1)})+\delta\hat{\mathfrak{V}}(x_{p_{n-1}}^{(n-1)})\Big)^{a_{n-1}a_{n-2}}_{M_{n-1};N_{n-2}}\;{}^{a_{n-2}}\widehat{\langle x_{p_{n-1}}^{(n-1)}}|\hat G^{(0)}_{N_{n-2};M_{n-2}}|\widehat{x_{p_{n-2}}^{(n-2)}\rangle}^{a_{n-2}}+$$
$$-\;\delta\hat{\mathcal{K}}^{\boldsymbol{\lambda};a_{n-1}a_{n-2}}_{M_{n-1};N_{n-2}}(x_{p_{n-1}}^{(n-1)})\;\Big({}^{a_{n-2}}\widehat{\langle x_{p_{n-1}}^{(n-1)}}|\hat G^{(0)}\;\big(\hat\partial_{p,\boldsymbol{\lambda}}\hat{\mathfrak{V}}\big)\;\hat G^{(0)}|\widehat{x_{p_{n-2}}^{(n-2)}\rangle}^{a_{n-2}}\Big)_{N_{n-2};M_{n-2}}+$$

$$
\begin{aligned}
+ &\quad \delta\hat{\mathfrak{K}}^{\boldsymbol{\lambda};a_{n-1}a_{n-2}}_{M_{n-1};N_{n-2}}(x^{(n-1)}_{p_{n-1}})\ {}^{a_{n-2}}\langle\widehat{x^{(n-1)}_{p_{n-1}}}|\hat{G}^{(0)}_{N_{n-2};M_{n-2}}|\widehat{x^{(n-2)}_{p_{n-2}}}\rangle^{a_{n-2}}\ \hat{\boldsymbol{\partial}}^{\boldsymbol{(n-2)}}_{\boldsymbol{p_{n-2},\lambda}}\Big] \times \\
\times &\quad \ldots\ldots \times \\
\times &\quad \Big[\Big(\hat{\mathfrak{k}}(x^{(2)}_{p_2}) + \delta\hat{\mathfrak{V}}(x^{(2)}_{p_2})\Big)^{a_2a_1}_{M_2;N_1}\ {}^{a_1}\langle\widehat{x^{(2)}_{p_2}}|\hat{G}^{(0)}_{N_1;M_1}|\widehat{x^{(1)}_{p_1}}\rangle^{a_1} + \\
- &\quad \delta\hat{\mathfrak{K}}^{\boldsymbol{\mu};a_2a_1}_{M_2;N_1}(x^{(2)}_{p_2})\ \Big({}^{a_1}\langle\widehat{x^{(2)}_{p_2}}|\hat{G}^{(0)}\ \big(\hat{\partial}_{p,\boldsymbol{\mu}}\hat{\mathfrak{V}}\big)\ \hat{G}^{(0)}|\widehat{x^{(1)}_{p_1}}\rangle^{a_1}\Big)_{N_1;M_1} + \\
+ &\quad \delta\hat{\mathfrak{K}}^{\boldsymbol{\mu};a_2a_1}_{M_2;N_1}(x^{(2)}_{p_2})\ {}^{a_1}\langle\widehat{x^{(2)}_{p_2}}|\hat{G}^{(0)}_{N_1;M_1}|\widehat{x^{(1)}_{p_1}}\rangle^{a_1}\ \hat{\boldsymbol{\partial}}^{\boldsymbol{(1)}}_{\boldsymbol{p_1,\mu}}\Big] \times \\
\times &\quad \Big[\Big(\hat{\mathfrak{k}}(x^{(1)}_{p_1}) + \delta\hat{\mathfrak{V}}(x^{(1)}_{p_1})\Big)^{a_1a_0}_{M_1;N_0}\ {}^{a_0}\langle\widehat{x^{(1)}_{p_1}}|\hat{G}^{(0)}_{N_0;M_0}|\widehat{x^{(0)}_{p_0}}\rangle^{a_0} + \\
- &\quad \delta\hat{\mathfrak{K}}^{\boldsymbol{\nu};a_1a_0}_{M_1;N_0}(x^{(1)}_{p_1})\ \Big({}^{a_0}\langle\widehat{x^{(1)}_{p_1}}|\hat{G}^{(0)}\ \big(\hat{\partial}_{p,\boldsymbol{\nu}}\hat{\mathfrak{V}}\big)\ \hat{G}^{(0)}|\widehat{x^{(0)}_{p_0}}\rangle^{a_0}\Big)_{N_0;M_0} + \\
+ &\quad \delta\hat{\mathfrak{K}}^{\boldsymbol{\nu};a_1a_0}_{M_1;N_0}(x^{(1)}_{p_1})\ {}^{a_0}\langle\widehat{x^{(1)}_{p_1}}|\hat{G}^{(0)}_{N_0;M_0}|\widehat{x^{(0)}_{p_0}}\rangle^{a_0}\ \hat{\boldsymbol{\partial}}^{\boldsymbol{(0)}}_{\boldsymbol{p_0,\nu}}\Big]\ ,\hat{S}\Big\}_+\ \mathfrak{W}^{(\bar{f},\bar{m},\bar{r});a_0}_{f_0,m_0,r_0}(x^{(0)}_{p_0};k_p)\Big\rangle_{\hat{\boldsymbol{\mathcal{V}}}}\ ; \\
&\quad \big(\text{summations in (D.52) without } \ a_n = a_{n-1} = \ldots = a_1 = a_0\ !\big)\ .
\end{aligned}
$$

Due to Derrick's theorem [13], we analyze the orders up to four gradients in (D.52) for a final effective Lagrangian. As one remembers relation (D.41), we have to exclude the completely diagonal term of the anomalous doubling '$a_n = a_{n-1} = \ldots = a_1 = a_0$' in (D.52) because this term only contributes vanishing measure in the spacetime integrations due to contradictory propagations of the generalized Heaviside functions. The first order term $n = 1$ (D.53) is not effected by this vanishing term of contradictory propagations and simply reduces to the part $\delta\hat{\mathfrak{k}}(x_p) = \hat{T}^{-1}(x_p)\,\hat{S}\,(\,\hat{\beta}\,\hat{\not{\partial}}_p\hat{T}(x_p)\,)$ (D.7) with saturated derivatives because the unsaturated derivative operators $\hat{\boldsymbol{\partial}}_{\boldsymbol{p,\kappa}}$ with $\delta\hat{\mathfrak{K}}^{\boldsymbol{\kappa}}(x_p)$ only lead to vanishing four-momentum integrals due to an anti-symmetric integrand. However, the commutators $[\hat{\boldsymbol{\partial}}_{\boldsymbol{p,\kappa}}\boldsymbol{,}\hat{G}^{(0)}]_-$ (D.31) are involved in every factor of $(\,\Delta\hat{\mathfrak{H}}\ \hat{G}^{(0)}\,)^n$ at order 'n' where the unsaturated gradient $\hat{\boldsymbol{\partial}}_{\boldsymbol{p,\kappa}}$ acts on the background potentials $\mathcal{V}_{\alpha;\kappa_1}(x_p)$ within the Green functions. This causes the additional part with '$-\langle(\,\hat{\partial}_{p,\kappa}\mathcal{V}_{\alpha;\kappa_1}(x_p)\,)\rangle_{\hat{\mathcal{V}}}\ \delta\hat{\mathfrak{K}}^{\boldsymbol{\kappa}}(x_p)\ \hat{V}^{\kappa_1}_{\alpha}$' at first order 'n=1'

$$
\begin{aligned}
&\Big\langle\mathcal{A}_{DET}\big[\hat{T},\hat{\mathcal{V}};\hat{\mathcal{J}}\equiv 0;n=1\big]\Big\rangle_{\hat{\boldsymbol{\mathcal{V}}}} = k^4_{max}\ \mathcal{N}_k\ \mathcal{N}\ \frac{1}{2}\ \int_C d^4x_p\ \times \qquad\qquad (D.53)\\
&\times\ \Big\langle\underset{N_f,\hat{\gamma}^{(\mu)}_{mn},N_c}{\overset{a(=1,2)}{\mathfrak{Tr}}}\Big[\frac{\hat{S}}{2}\Big\{\Big(\delta\hat{\mathfrak{k}}(x_p) - \big(\hat{\partial}_{p,\kappa}\mathcal{V}_{\alpha;\kappa_1}(x_p)\big)\ \delta\hat{\mathfrak{K}}^{\boldsymbol{\kappa}}(x_p)\ \hat{V}^{\kappa_1}_{\alpha}\Big)\ ,\hat{S}\Big\}_+\Big]\Big\rangle_{\hat{\boldsymbol{\mathcal{V}}}} = k^4_{max}\ \mathcal{N}_k\ \mathcal{N}\ \frac{1}{2}\ \int_C d^4x_p\ \times \\
&\times\ \Big(\underset{N_f,\hat{\gamma}^{(\mu)}_{mn},N_c}{\overset{a(=1,2)}{\mathfrak{Tr}}}\Big[\hat{T}^{-1}(x_p)\ \hat{S}\ \big(\hat{\beta}\ \hat{\not{\partial}}_p\hat{T}(x_p)\big)\Big] - \Big\langle\big(\hat{\partial}_{p,\kappa}\mathcal{V}_{\alpha;\kappa_1}(x_p)\big)\Big\rangle_{\hat{\boldsymbol{\mathcal{V}}}}\ \underset{N_f,\hat{\gamma}^{(\mu)}_{mn},N_c}{\overset{a(=1,2)}{\mathfrak{Tr}}}\Big[\hat{T}^{-1}(x_p)\ \hat{S}\ \hat{\beta}\ \hat{\gamma}^{\kappa}\ \hat{T}(x_p)\ \hat{V}^{\kappa_1}_{\alpha}\Big]\Big)\ .
\end{aligned}
$$

The remaining terms in (D.53) contain the cutoff momentum k^4_{max} to the power of four as a scale for other gradually varying gradients. As we proceed to higher order gradients as $n = 2$, one also finds terms with k^6_{max} which can be neglected in a lowest order momentum-energy expansion (with $k_{max} \ll 1$ in dimensionless units).

The second order term $n = 2$ (D.54) is calculated with detailed description of the various steps. In order to exclude vanishing contradictory propagation of $\hat{G}^{(0)}$ with $a_2 = a_1 = a_0$, one has to choose the two commutator terms (D.40) for the off-diagonal blocks $\Delta\hat{\mathfrak{H}}^{a_2\neq a_1}_{M_2;M_1}(x_p)$, $\Delta\hat{\mathfrak{H}}^{a_1\neq a_0}_{M_1;M_0}(x_p)$ in the BCS-sector of the anomalous doubled space. We substitute the total gradient operators $\Delta\hat{\mathfrak{H}}(x_p)$ by the part $\delta\hat{\mathfrak{h}}(x_p)$ (D.7) with unsaturated derivatives and potential matrix $\hat{\mathfrak{V}}(x_p)$ (D.8) and by the term $\delta\hat{\mathfrak{K}}^{\boldsymbol{\mu}}(x_p)\ \hat{\boldsymbol{\partial}}_{\boldsymbol{p,\mu}}$ with unsaturated gradient operators. The latter act onto the anomalous doubled plane wave states of four-momentum k_p. We also incorporate the commutator (D.32) and the relations (D.33-D.38) as

abbreviating symbols in the second order term '$n=2$' (D.54) with application of propagation rules (D.25-D.28)

$$\begin{aligned}
&\Big\langle\mathcal{A}_{DET}\big[\hat{T},\hat{\mathcal{V}};\hat{\mathcal{J}}\equiv 0;n=2\big]\Big\rangle_{\hat{\mathcal{V}}}=\sum_{p=\pm}\int^{|k_p|<k_{max}/2} d^4k_p\;\mathcal{N}_k\;\mathcal{N}\sum_{\bar{M}}\Big(-\frac{1}{4}\Big)\int_{-\infty}^{+\infty}d^4x_p\;\eta_p\;\times \qquad\text{(D.54)}\\
&\times\;\Bigg\langle\Big(\mathfrak{W}_{M_2}^{(\bar{M});a_2}(x_p;k_p)\Big)^{\dagger}\frac{\hat{S}}{2}\Bigg\{\frac{\hat{S}}{2}\Big[\Delta\hat{\mathfrak{h}}(x_p)\,,\,\hat{S}\Big]_{-}\frac{\hat{S}}{2}\Big[\Delta\hat{\mathfrak{h}}(x_p)\,,\,\hat{S}\Big]_{-}+\frac{\hat{S}}{2}\Big[\delta\hat{\mathfrak{K}}^{\boldsymbol{\kappa}}(x_p)\,,\,\hat{S}\Big]_{-}\times\\
&\times\;\frac{\hat{S}}{2}\Big[\Big(\hat{\partial}_{p,\boldsymbol{\kappa}}\Delta\hat{\mathfrak{h}}(x_p)\Big)-\delta\hat{\mathfrak{h}}(x_p)\big(\hat{\partial}_{p,\boldsymbol{\kappa}}\hat{\mathfrak{V}}(x_p)\big)+\\
&+\;\delta\hat{\mathfrak{K}}^{\boldsymbol{\lambda}}(x_p)\Big\{\big(\hat{\partial}_{p,\boldsymbol{\kappa}}\hat{\mathfrak{V}}(x_p)\big)\,,\,\big(\hat{\partial}_{p,\boldsymbol{\lambda}}\hat{\mathfrak{V}}(x_p)\big)\Big\}_{+}\,,\,\hat{S}\Big]_{-}\,,\,\hat{S}\Bigg\}_{+;M_2;M_0}^{a_2=a_0}\mathfrak{W}_{M_0}^{(\bar{M});a_0}(x_p;k_p)+\\
&+\;\Big(\mathfrak{W}_{M_2}^{(\bar{M});a_2}(x_p;k_p)\Big)^{\dagger}\frac{\hat{S}}{2}\Bigg\{\frac{\hat{S}}{2}\Big[\delta\hat{\mathfrak{K}}^{\boldsymbol{\kappa}}(x_p)\,,\,\hat{S}\Big]_{-}\frac{\hat{S}}{2}\Big[\delta\hat{\mathfrak{K}}^{\boldsymbol{\lambda}}(x_p)\,,\,\hat{S}\Big]_{-}\,,\,\hat{S}\Bigg\}_{+;M_2;M_0}^{a_2=a_0}\hat{\boldsymbol{\partial}}_{\boldsymbol{p},\boldsymbol{\kappa}}\hat{\boldsymbol{\partial}}_{\boldsymbol{p},\boldsymbol{\lambda}}\mathfrak{W}_{M_0}^{(\bar{M});a_0}(x_p;k_p)+\\
&+\;\text{'linear' 'unsaturated' gradient operator terms}\Bigg\rangle_{\hat{\mathcal{V}}}\;.
\end{aligned}$$

Since the four-momentum integrals vanish for the case with linear, anti-symmetric integrand, one only has an additional integration of quadratic order $k_{p,\kappa}\,k_{p,\lambda}$ from the action of unsaturated gradient operators onto plane wave states

$$\begin{aligned}
&\Big\langle\mathcal{A}_{DET}\big[\hat{T},\hat{\mathcal{V}};\hat{\mathcal{J}}\equiv 0;n=2\big]\Big\rangle_{\hat{\mathcal{V}}}=\sum_{p=\pm}\int^{|k_p|<k_{max}/2} d^4k_p\;\mathcal{N}_k\;\mathcal{N}\sum_{\bar{M}}\Big(-\frac{1}{4}\Big)\int_{-\infty}^{+\infty}d^4x_p\;\eta_p\;\times \qquad\text{(D.55)}\\
&\times\;\Bigg\langle\Big(\mathfrak{W}_{M_2}^{(\bar{M});a_2}(x_p;k_p)\Big)^{\dagger}\frac{\hat{S}}{2}\Bigg\{\frac{\hat{S}}{2}\Big[\Delta\hat{\mathfrak{h}}(x_p)\,,\,\hat{S}\Big]_{-}\frac{\hat{S}}{2}\Big[\Delta\hat{\mathfrak{h}}(x_p)\,,\,\hat{S}\Big]_{-}+\frac{\hat{S}}{2}\Big[\delta\hat{\mathfrak{K}}^{\boldsymbol{\kappa}}(x_p)\,,\,\hat{S}\Big]_{-}\times\\
&\times\;\frac{\hat{S}}{2}\Big[\Big(\hat{\partial}_{p,\boldsymbol{\kappa}}\Delta\hat{\mathfrak{h}}(x_p)\Big)-\delta\hat{\mathfrak{h}}(x_p)\big(\hat{\partial}_{p,\boldsymbol{\kappa}}\hat{\mathfrak{V}}(x_p)\big)+\\
&+\;\delta\hat{\mathfrak{K}}^{\boldsymbol{\lambda}}(x_p)\Big\{\big(\hat{\partial}_{p,\boldsymbol{\kappa}}\hat{\mathfrak{V}}(x_p)\big)\,,\,\big(\hat{\partial}_{p,\boldsymbol{\lambda}}\hat{\mathfrak{V}}(x_p)\big)\Big\}_{+}\,,\,\hat{S}\Big]_{-}\,,\,\hat{S}\Bigg\}_{+;M_2;M_0}^{a_2=a_0}\mathfrak{W}_{M_0}^{(\bar{M});a_0}(x_p;k_p)+\Big(\mathfrak{W}_{M_2}^{(\bar{M});a_2}(x_p;k_p)\Big)^{\dagger}\times\\
&\times\;\frac{\hat{S}}{2}\Bigg\{\frac{\hat{S}}{2}\Big[\delta\hat{\mathfrak{K}}^{\boldsymbol{\kappa}}(x_p)\,,\,\hat{S}\Big]_{-}\frac{\hat{S}}{2}\Big[\delta\hat{\mathfrak{K}}^{\boldsymbol{\lambda}}(x_p)\,,\,\hat{S}\Big]_{-}\,,\,\hat{S}\Bigg\}_{+;M_2;M_0}^{a_2=a_0}\Big(\imath\,k_{p,\kappa}\;\imath\,k_{p,\lambda}\Big)\;\hat{S}^2\;\mathfrak{W}_{M_0}^{(\bar{M});a_0}(x_p;k_p)\Bigg\rangle_{\hat{\mathcal{V}}}\;.
\end{aligned}$$

Apart from the power of four term k_{max}^4 for the parts without action onto plane waves, one therefore attains an additional order of $k_{max}^6/12$ for the part following from unsaturated gradient operator actions onto plane wave states

$$\sum_{p=\pm}\int^{|k_p|<k_{max}/2} d^4k_p\;\Big(\imath\,k_{p,\kappa}\;\imath\,k_{p,\lambda}\Big)=-\eta_{\kappa\lambda}\;\frac{k_{max}^6}{12}\;\sum_{p=\pm}\;. \qquad\text{(D.56)}$$

After insertion of (D.56) into (D.55), we obtain relation (D.57) with additional, relative order $k_{max}^2/12$ for the parts resulting from the unsaturated gradient operator action onto $\mathfrak{W}_{M_0}^{(\bar{M});a_0}(x_p;k_p)$. As already stated, we assume gradually varying BCS-terms in the coset matrices $\hat{T}(x_p)=\exp\{-\hat{Y}(x_p)\}$ with $k_{max}\ll 1$ in dimensionless units

$$\Big\langle\mathcal{A}_{DET}\big[\hat{T},\hat{\mathcal{V}};\hat{\mathcal{J}}\equiv 0;n=2\big]\Big\rangle_{\hat{\mathcal{V}}}=k_{max}^4\;\mathcal{N}_k\;\mathcal{N}\;\Big(-\frac{1}{4}\Big)\int_C d^4x_p\;\times \qquad\text{(D.57)}$$

$$
\begin{aligned}
\times\; & \Bigg\{\Bigg\langle \mathop{\mathfrak{Tr}}_{N_f,\hat{\gamma}_{mn}^{(\mu)},N_c}^{a(=1,2)}\Bigg[\frac{\hat{S}}{2}\Bigg\{\frac{\hat{S}}{2}\Big[\Delta\hat{\mathfrak{h}}(x_p)\,,\,\hat{S}\Big]_-\frac{\hat{S}}{2}\Big[\Delta\hat{\mathfrak{h}}(x_p)\,,\,\hat{S}\Big]_-\,,\,\hat{S}\Bigg\}_+ + \\
+\; & \frac{\hat{S}}{2}\Bigg\{\frac{\hat{S}}{2}\Big[\delta\hat{\mathcal{K}}^{\boldsymbol{\kappa}}(x_p)\,,\,\hat{S}\Big]_-\frac{\hat{S}}{2}\Big[\Big(\hat{\partial}_{p,\boldsymbol{\kappa}}\Delta\hat{\mathfrak{h}}(x_p)\Big) - \delta\hat{\mathfrak{h}}(x_p)\,\big(\hat{\partial}_{p,\boldsymbol{\kappa}}\hat{\mathfrak{V}}(x_p)\big) + \\
+\; & \delta\hat{\mathcal{K}}^{\boldsymbol{\lambda}}(x_p)\,\Big\{\big(\hat{\partial}_{p,\boldsymbol{\kappa}}\hat{\mathfrak{V}}(x_p)\big)\,,\,\big(\hat{\partial}_{p,\boldsymbol{\lambda}}\hat{\mathfrak{V}}(x_p)\big)\Big\}_+\,,\,\hat{S}\Big]_-\,,\,\hat{S}\Bigg\}_+\Bigg]\Bigg\rangle_{\hat{\mathcal{V}}} \\
-\; & \eta_{\kappa\lambda}\,\frac{k_{max}^2}{12}\mathop{\mathfrak{Tr}}_{N_f,\hat{\gamma}_{mn}^{(\mu)},N_c}^{a(=1,2)}\Bigg[\frac{\hat{S}}{2}\Bigg\{\frac{\hat{S}}{2}\Big[\delta\hat{\mathcal{K}}^{\boldsymbol{\kappa}}(x_p)\,,\,\hat{S}\Big]_-\frac{\hat{S}}{2}\Big[\delta\hat{\mathcal{K}}^{\boldsymbol{\lambda}}(x_p)\,,\,\hat{S}\Big]_-\,,\,\hat{S}\Bigg\}_+\Bigg]\Bigg\}\;;
\end{aligned}
$$

$$
k_{max} \ll 1 \quad \text{dimensionless units !}\,. \tag{D.58}
$$

This assumption allows to simplify the second order term (D.57) to (D.59) without action of unsaturated gradient operators onto plane wave states

$$
\begin{aligned}
\Big\langle \mathcal{A}_{DET}\big[\hat{T},\hat{\mathcal{V}};\hat{\mathcal{J}}\equiv 0;n=2\big]\Big\rangle_{\hat{\mathcal{V}}} &= k_{max}^4\,\mathcal{N}_k\,\mathcal{N}\,\Big(-\frac{1}{4}\Big)\int_C d^4x_p\,\times \\
\times\; & \Bigg\langle \mathop{\mathfrak{Tr}}_{N_f,\hat{\gamma}_{mn}^{(\mu)},N_c}^{a(=1,2)}\Bigg[\frac{\hat{S}}{2}\Bigg\{\frac{\hat{S}}{2}\Big[\Delta\hat{\mathfrak{h}}(x_p)\,,\,\hat{S}\Big]_-\frac{\hat{S}}{2}\Big[\Delta\hat{\mathfrak{h}}(x_p)\,,\,\hat{S}\Big]_-\,,\,\hat{S}\Bigg\}_+ + \\
+\; & \frac{\hat{S}}{2}\Bigg\{\frac{\hat{S}}{2}\Big[\delta\hat{\mathcal{K}}^{\boldsymbol{\kappa}}(x_p)\,,\,\hat{S}\Big]_-\frac{\hat{S}}{2}\Big[\Big(\hat{\partial}_{p,\boldsymbol{\kappa}}\Delta\hat{\mathfrak{h}}(x_p)\Big) - \delta\hat{\mathfrak{h}}(x_p)\,\big(\hat{\partial}_{p,\boldsymbol{\kappa}}\hat{\mathfrak{V}}(x_p)\big) + \\
+\; & \delta\hat{\mathcal{K}}^{\boldsymbol{\lambda}}(x_p)\,\Big\{\big(\hat{\partial}_{p,\boldsymbol{\kappa}}\hat{\mathfrak{V}}(x_p)\big)\,,\,\big(\hat{\partial}_{p,\boldsymbol{\lambda}}\hat{\mathfrak{V}}(x_p)\big)\Big\}_+\,,\,\hat{S}\Big]_-\,,\,\hat{S}\Bigg\}_+\Bigg]\Bigg\rangle_{\hat{\mathcal{V}}}\,.
\end{aligned} \tag{D.59}
$$

The anti-commutator $\frac{1}{2}\hat{S}\,\{\ldots\,,\hat{S}\}_+$ (D.44) for $\delta_{a_2a_0}$ can be removed in (D.59) so that the second order term finally reduces to (D.60) because the traces of (D.59) regard only terms $a_2=a_0$ in any case of values for a_2, a_0

$$
\begin{aligned}
\Big\langle \mathcal{A}_{DET}\big[\hat{T},\hat{\mathcal{V}};\hat{\mathcal{J}}\equiv 0;n=2\big]\Big\rangle_{\hat{\mathcal{V}}} &= k_{max}^4\,\mathcal{N}_k\,\mathcal{N}\,\Big(-\frac{1}{4}\Big)\int_C d^4x_p\,\times \\
\times\; & \Bigg\langle \mathop{\mathfrak{Tr}}_{N_f,\hat{\gamma}_{mn}^{(\mu)},N_c}^{a(=1,2)}\Bigg[\frac{\hat{S}}{2}\Big[\Delta\hat{\mathfrak{h}}(x_p)\,,\,\hat{S}\Big]_-\frac{\hat{S}}{2}\Big[\Delta\hat{\mathfrak{h}}(x_p)\,,\,\hat{S}\Big]_-\Bigg] + \mathop{\mathfrak{Tr}}_{N_f,\hat{\gamma}_{mn}^{(\mu)},N_c}^{a(=1,2)}\Bigg[\frac{\hat{S}}{2}\Big[\delta\hat{\mathcal{K}}^{\boldsymbol{\kappa}}(x_p)\,,\,\hat{S}\Big]_-\times \\
\times\; & \frac{\hat{S}}{2}\Big[\Big(\hat{\partial}_{p,\boldsymbol{\kappa}}\Delta\hat{\mathfrak{h}}(x_p)\Big) - \delta\hat{\mathfrak{h}}(x_p)\,\big(\hat{\partial}_{p,\boldsymbol{\kappa}}\hat{\mathfrak{V}}(x_p)\big) + \delta\hat{\mathcal{K}}^{\boldsymbol{\lambda}}(x_p)\,\Big\{\big(\hat{\partial}_{p,\boldsymbol{\kappa}}\hat{\mathfrak{V}}(x_p)\big)\,,\,\big(\hat{\partial}_{p,\boldsymbol{\lambda}}\hat{\mathfrak{V}}(x_p)\big)\Big\}_+\,,\,\hat{S}\Big]_-\Bigg]\Bigg\rangle_{\hat{\mathcal{V}}} \\
\approx\; & k_{max}^4\,\mathcal{N}_k\,\mathcal{N}\,\Big(-\frac{1}{4}\Big)\int_C d^4x_p\;\times \\
\times\; & \Bigg\langle \mathop{\mathfrak{Tr}}_{N_f,\hat{\gamma}_{mn}^{(\mu)},N_c}^{a(=1,2)}\Bigg[\frac{\hat{S}}{2}\Big[\delta\hat{\mathfrak{h}}(x_p)\,,\,\hat{S}\Big]_-\frac{\hat{S}}{2}\Big[\delta\hat{\mathfrak{h}}(x_p)\,,\,\hat{S}\Big]_- + \frac{\hat{S}}{2}\Big[\delta\hat{\mathcal{K}}^{\boldsymbol{\kappa}}(x_p)\,,\,\hat{S}\Big]_-\frac{\hat{S}}{2}\Big[\Big(\hat{\partial}_{p,\boldsymbol{\kappa}}\delta\hat{\mathfrak{h}}(x_p)\Big)\,,\,\hat{S}\Big]_-\Bigg]\Bigg\rangle_{\hat{\mathcal{V}}}\,.
\end{aligned} \tag{D.60}
$$

If one disregards surface phenomena at the boundary of the nucleus, we can further assume a constant effective background potential $\hat{\mathcal{V}}(x_p)$ so that one can reduce the second order gradient expansion (D.60) to the simplified relation in the last two lines of (D.60) for the bulk of a nucleus.

In the remainder we restrict to the case of constant background potentials $\hat{\mathcal{V}}(x_p)$ so that we can eventually specify the third order term (D.61) of gradients. Since the completely block-diagonal propagation of Green functions $\hat{G}^{(0)}$ vanishes for $\mathcal{A}_{DET}[\hat{T},\hat{\mathcal{V}};\hat{\mathcal{J}}\equiv 0]\rangle_{\hat{\mathcal{V}}}$ due to contradictory propagations with time contour extended Heaviside functions, one has to consider two off-diagonal terms $\Delta\hat{\mathcal{H}}^{a_3\neq a_2}(x_p)$, $\Delta\hat{\mathcal{H}}^{a_2\neq a_1}(x_p)$ and one block-diagonal gradient operator $\Delta\hat{\mathcal{H}}^{a_1=a_0}(x_p)$. According to (D.39,D.40), this amounts to two commutator terms and one anti-commutator part. Since there are three possibilities of combinatorial ordering of two commutators and one anti-commutator terms, one has to start from relation (D.61) for the third order part $n=3$ with additional, combinatorial factor '3'

$$\Big\langle\mathcal{A}_{DET}\big[\hat{T},\hat{\mathcal{V}};\hat{\mathcal{J}}\equiv 0;n=3\big]\Big\rangle_{\hat{\mathcal{V}}}=\sum_{p=\pm}\int^{|k_p|<k_{max}/2}d^4k_p\;\mathcal{N}_k\;\mathcal{N}\sum_{\bar{M}}3\;\frac{1}{2}\;\frac{1}{3}\int_{-\infty}^{+\infty}d^4x_p\;\eta_p\;\times \tag{D.61}$$
$$\times\;\;\bigg\langle\Big(\mathfrak{W}_{M_3}^{(\bar{M});a_3}(x_p;k_p)\Big)^\dagger\frac{\hat{S}}{2}\bigg\{\frac{\hat{S}}{2}\Big[\Delta\hat{\mathcal{H}}_{M_3;M_2}^{a_3\neq a_2}(x_p)\;,\;\hat{S}\Big]_-\frac{\hat{S}}{2}\Big[\Delta\hat{\mathcal{H}}_{M_2;M_1}^{a_2\neq a_1}(x_p)\;,\;\hat{S}\Big]_-\;\times$$
$$\times\;\;\frac{\hat{S}}{2}\Big\{\Delta\hat{\mathcal{H}}_{M_1;M_0}^{a_1=a_0}(x_p)\;,\;\hat{S}\Big\}_+\;,\;\hat{S}\bigg\}_+\mathfrak{W}_{M_0}^{(\bar{M});a_0}(x_p;k_p)\bigg\rangle_{\hat{\mathcal{V}}}\;.$$

Equation (D.62) follows from further transformation to the parts $\delta\hat{\mathfrak{h}}(x_p)$, $\delta\hat{\mathcal{K}}^{\boldsymbol{\mu}}(x_p)\;\hat{\boldsymbol{\partial}}_{\boldsymbol{p,\mu}}$ with saturated and unsaturated derivatives where we only keep the order k_{max}^4 and neglect higher orders of $k_{max}^{\geq 6}$ following from action of unsaturated gradient operators onto plane wave states

$$\Big\langle\mathcal{A}_{DET}\big[\hat{T},\hat{\mathcal{V}};\hat{\mathcal{J}}\equiv 0;n=3\big]\Big\rangle_{\hat{\mathcal{V}}}\approx k_{max}^4\;\mathcal{N}_k\;\mathcal{N}\;\frac{1}{2}\int_C d^4x_p\;\times \tag{D.62}$$
$$\times\;\;\Bigg[\Bigg\langle\underset{N_f,\hat{\gamma}_{mn}^{(\mu)},N_c}{\overset{a(=1,2)}{\mathfrak{Tr}}}\Bigg[\frac{\hat{S}}{2}\bigg\{\Big(\frac{\hat{S}}{2}\Big[\delta\hat{\mathfrak{h}}(x_p)\;,\;\hat{S}\Big]_-\Big)^2\frac{\hat{S}}{2}\Big\{\delta\hat{\mathfrak{h}}(x_p)\;,\;\hat{S}\Big\}_+\;,\;\hat{S}\bigg\}_+\Bigg]\Bigg\rangle_{\hat{\mathcal{V}}}+$$
$$+\;\;\Bigg\langle\underset{N_f,\hat{\gamma}_{mn}^{(\mu)},N_c}{\overset{a(=1,2)}{\mathfrak{Tr}}}\Bigg[\frac{\hat{S}}{2}\bigg\{\frac{\hat{S}}{2}\Big[\delta\hat{\mathfrak{h}}(x_p)\;,\;\hat{S}\Big]_-\frac{\hat{S}}{2}\Big[\delta\hat{\mathcal{K}}^{\boldsymbol{\mu}}(x_p)\;,\;\hat{S}\Big]_-\frac{\hat{S}}{2}\Big[\Big(\hat{\partial}_{p,\mu}\delta\hat{\mathfrak{h}}(x_p)\Big)\;,\;\hat{S}\Big]_+\;,\;\hat{S}\bigg\}_+\Bigg]\Bigg\rangle_{\hat{\mathcal{V}}}+$$
$$+\;\;\Bigg\langle\underset{N_f,\hat{\gamma}_{mn}^{(\mu)},N_c}{\overset{a(=1,2)}{\mathfrak{Tr}}}\Bigg[\frac{\hat{S}}{2}\bigg\{\frac{\hat{S}}{2}\Big[\delta\hat{\mathcal{K}}^{\boldsymbol{\nu}}(x_p)\;,\;\hat{S}\Big]_-\Big(\hat{\partial}_{p,\nu}\frac{\hat{S}}{2}\Big[\delta\hat{\mathfrak{h}}(x_p)\;,\;\hat{S}\Big]_-\frac{\hat{S}}{2}\Big\{\delta\hat{\mathfrak{h}}(x_p)\;,\;\hat{S}\Big\}_+\Big)\;,\;\hat{S}\bigg\}_+\Bigg]\Bigg\rangle_{\hat{\mathcal{V}}}+$$
$$+\;\;\Bigg\langle\underset{N_f,\hat{\gamma}_{mn}^{(\mu)},N_c}{\overset{a(=1,2)}{\mathfrak{Tr}}}\Bigg[\frac{\hat{S}}{2}\bigg\{\frac{\hat{S}}{2}\Big[\delta\hat{\mathcal{K}}^{\boldsymbol{\nu}}(x_p)\;,\;\hat{S}\Big]_-\Big(\hat{\partial}_{p,\nu}\frac{\hat{S}}{2}\Big[\delta\hat{\mathcal{K}}^{\boldsymbol{\mu}}(x_p)\;,\;\hat{S}\Big]_-\frac{\hat{S}}{2}\Big\{\Big(\hat{\partial}_{p,\mu}\delta\hat{\mathfrak{h}}(x_p)\Big)\;,\;\hat{S}\Big\}_+\Big)\;,\;\hat{S}\bigg\}_+\Bigg]\Bigg\rangle_{\hat{\mathcal{V}}}\Bigg]\;.$$

The anti-commutator (D.44) for the Kronecker delta $\delta_{a_3a_0}$ in the anomalous doubled space is removed as in (D.59,D.60) because the traces of (D.62) already reduce to the diagonal parts $a_3=a_0$ in any case. Therefore, one accomplishes relation (D.63) for the third order gradient term under approximation of $k_{max}\ll 1$ with k_{max}^4 as the lowest order four-momentum scale

$$\Big\langle\mathcal{A}_{DET}\big[\hat{T},\hat{\mathcal{V}};\hat{\mathcal{J}}\equiv 0;n=3\big]\Big\rangle_{\hat{\mathcal{V}}}\approx k_{max}^4\;\mathcal{N}_k\;\mathcal{N}\;\frac{1}{2}\int_C d^4x_p\;\times \tag{D.63}$$
$$\times\;\;\Bigg[\Bigg\langle\underset{N_f,\hat{\gamma}_{mn}^{(\mu)},N_c}{\overset{a(=1,2)}{\mathfrak{Tr}}}\Bigg[\Big(\frac{\hat{S}}{2}\Big[\delta\hat{\mathfrak{h}}(x_p)\;,\;\hat{S}\Big]_-\Big)^2\frac{\hat{S}}{2}\Big\{\delta\hat{\mathfrak{h}}(x_p)\;,\;\hat{S}\Big\}_+\Bigg]\Bigg\rangle_{\hat{\mathcal{V}}}+$$
$$+\;\;\Bigg\langle\underset{N_f,\hat{\gamma}_{mn}^{(\mu)},N_c}{\overset{a(=1,2)}{\mathfrak{Tr}}}\Bigg[\frac{\hat{S}}{2}\Big[\delta\hat{\mathfrak{h}}(x_p)\;,\;\hat{S}\Big]_-\frac{\hat{S}}{2}\Big[\delta\hat{\mathcal{K}}^{\boldsymbol{\mu}}(x_p)\;,\;\hat{S}\Big]_-\frac{\hat{S}}{2}\Big[\Big(\hat{\partial}_{p,\mu}\delta\hat{\mathfrak{h}}(x_p)\Big)\;,\;\hat{S}\Big\}_+\Bigg]\Bigg\rangle_{\hat{\mathcal{V}}}+$$

$$+ \left\langle \underset{N_f,\hat{\gamma}_{mn}^{(\mu)},N_c}{\overset{a(=1,2)}{\mathfrak{Tr}}} \left[\frac{\hat{S}}{2}\Big[\delta\hat{\mathcal{K}}^{\boldsymbol{\nu}}(x_p)\,,\,\hat{S}\Big]_-\Big(\hat{\partial}_{p,\nu}\frac{\hat{S}}{2}\Big[\delta\hat{\mathfrak{h}}(x_p)\,,\,\hat{S}\Big]_-\frac{\hat{S}}{2}\Big\{\delta\hat{\mathfrak{h}}(x_p)\,,\,\hat{S}\Big\}_+\Big)\right]\right\rangle_{\hat{\boldsymbol{\mathcal{V}}}} +$$
$$+ \left\langle \underset{N_f,\hat{\gamma}_{mn}^{(\mu)},N_c}{\overset{a(=1,2)}{\mathfrak{Tr}}} \left[\frac{\hat{S}}{2}\Big[\delta\hat{\mathcal{K}}^{\boldsymbol{\nu}}(x_p)\,,\,\hat{S}\Big]_-\Big(\hat{\partial}_{p,\nu}\frac{\hat{S}}{2}\Big[\delta\hat{\mathcal{K}}^{\boldsymbol{\mu}}(x_p)\,,\,\hat{S}\Big]_-\frac{\hat{S}}{2}\Big\{\Big(\hat{\partial}_{p,\mu}\delta\hat{\mathfrak{h}}(x_p)\Big)\,,\,\hat{S}\Big\}_+\Big)\right]\right\rangle_{\hat{\boldsymbol{\mathcal{V}}}}\Bigg] .$$

The fourth order derivative part stabilizes the static energy configurations so that we have to extract from relation (D.52) all terms with four gradients (under the assumption of a constant background potential); but in addition, these gradient terms of order four encompass various combinatorial factors. In general the n-th order gradient term of $\langle\mathcal{A}_{DET}[\hat{T},\hat{\mathcal{V}};\hat{\mathcal{J}}\equiv 0;n]\rangle_{\hat{\boldsymbol{\mathcal{V}}}}$ with $2m$ off-diagonal operators '$-\frac{1}{2}\hat{S}\,[\Delta\hat{\mathcal{H}}_{M_{i+1};M_i}^{a_{i+1}\neq a_i}(x_p)\,,\,\hat{S}]_-$' and $n-2m$ block diagonal gradient parts '$\frac{1}{2}\hat{S}\,\boldsymbol{\{}\Delta\hat{\mathcal{H}}_{M_{j+1};M_j}^{a_{j+1}=a_j}(x_p)\,,\,\hat{S}\boldsymbol{\}}_+$' is included by the coefficient $C_{(2m)}^{(n)}$-times which is determined by the binomial coefficient $\binom{n}{2m}$. However, these $C_{(2m)}^{(n)}=\binom{n}{2m}$ combinations are also realized in various patterns under the trace operations so that we have to introduce combinatorial sub-factors $C_{(2m),l}^{(n)}$ whose sum $\sum_l C_{(2m),l}^{(n)}$ is equivalent to the total number of combinations $C_{(2m)}^{(n)}$ for the n-th order gradient term with $2m$ gradients in the BCS- or off-diagonal, anomalous-doubled sectors. Since one has to exclude the completely block diagonal terms according to opposite propagations of Green functions, we have one combination $C_{(2m=4)}^{(n=4)}=\binom{4}{4}=1$ for four off-diagonal parts $\Delta\hat{\mathcal{H}}_{M_{i+1};M_i}^{a_{i+1}\neq a_i}(x_p)$ and six combinations $C_{(2m=2)}^{(n=4)}=\binom{4}{2}=6$ for the case of two BCS-sector parts and two block diagonal terms. The latter six combinations differ by the pattern (D.64) with $C_{(2m=2),l=1}^{(n=4)}=2$ realizations under the trace

$$\Delta\hat{\mathcal{H}}_{M_4;M_3}^{a_4\neq a_3}(x_p)\quad\Delta\hat{\mathcal{H}}_{M_3;M_2}^{a_3=a_2}(x_p)\quad\Delta\hat{\mathcal{H}}_{M_2;M_1}^{a_2\neq a_1}(x_p)\quad\Delta\hat{\mathcal{H}}_{M_1;M_0}^{a_1=a_0}(x_p)\;, \tag{D.64}$$

and four combinations $C_{(2m=2),l=2}^{(n=4)}=4$ for the pattern (D.65)

$$\Delta\hat{\mathcal{H}}_{M_4;M_3}^{a_4\neq a_3}(x_p)\quad\Delta\hat{\mathcal{H}}_{M_3;M_2}^{a_3\neq a_2}(x_p)\quad\Delta\hat{\mathcal{H}}_{M_2;M_1}^{a_2=a_1}(x_p)\quad\Delta\hat{\mathcal{H}}_{M_1;M_0}^{a_1=a_0}(x_p)\;, \tag{D.65}$$

yielding the total number of six combinations $C_{(2m=2)}^{(n=4)}=\binom{4}{2}=6$

$$\sum_{l=1,2}C_{(2m=2),l}^{(n=4)}=\underbrace{C_{(2m=2),l=1}^{(n=4)}}_{=2}+\underbrace{C_{(2m=2),l=2}^{(n=4)}}_{=4}=C_{(2m=2)}^{(n=4)}=\binom{4}{2}=6\;. \tag{D.66}$$

The various combinations with different sub-patterns for the ordering of block diagonal density parts and BCS-sectors therefore comprise the corresponding anti-commutator '+' and commutator '-' operators (D.39,D.40) which we abbreviate by the common symbol $[\ldots\;,\;\ldots\boldsymbol{\}}_{\boldsymbol{I,J,K,L=\pm}}$ with indices $I,J,K,L=\pm$ specifying the anti-commutator '+' or commutator '-' instead of the symbols $\boldsymbol{\{}\ldots\;,\;\ldots\boldsymbol{\}}_{\boldsymbol{I,J,K,L=+}}$ (D.39) and $[\ldots\;,\;\ldots]_{\boldsymbol{I,J,K,L=-}}$ (D.40), respectively

$$\Big\langle\mathcal{A}_{DET}\big[\hat{T},\hat{\mathcal{V}};\hat{\mathcal{J}}\equiv 0;n=4\big]\Big\rangle_{\hat{\boldsymbol{\mathcal{V}}}}\approx k_{max}^4\,\mathcal{N}_k\,\mathcal{N}\left(-\frac{1}{8}\right)\int_C d^4x_p\sum_{m=1,2}\sum_{l=1}^{3-m}C_{(2m),l}^{(n=4)}\times \tag{D.67}$$
$$\times\left\langle\underset{N_f,\hat{\gamma}_{mn}^{(\mu)},N_c}{\overset{a(=1,2)}{\mathfrak{Tr}}}\left[\frac{\hat{S}}{2}\Big[\delta\hat{\mathfrak{h}}(x_p)\,,\,\hat{S}\Big\}_{\boldsymbol{I_{m_l}}}\frac{\hat{S}}{2}\Big[\delta\hat{\mathfrak{h}}(x_p)\,,\,\hat{S}\Big\}_{\boldsymbol{J_{m_l}}}\frac{\hat{S}}{2}\Big[\delta\hat{\mathfrak{h}}(x_p)\,,\,\hat{S}\Big\}_{\boldsymbol{K_{m_l}}}\frac{\hat{S}}{2}\Big[\delta\hat{\mathfrak{h}}(x_p)\,,\,\hat{S}\Big\}_{\boldsymbol{L_{m_l}}}+\right.\right.$$
$$+\frac{\hat{S}}{2}\Big[\delta\hat{\mathfrak{h}}(x_p)\,,\,\hat{S}\Big\}_{\boldsymbol{I_{m_l}}}\frac{\hat{S}}{2}\Big[\delta\hat{\mathfrak{h}}(x_p)\,,\,\hat{S}\Big\}_{\boldsymbol{J_{m_l}}}\frac{\hat{S}}{2}\Big[\delta\hat{\mathcal{K}}^{\boldsymbol{\lambda}}(x_p)\,,\,\hat{S}\Big\}_{\boldsymbol{K_{m_l}}}\Big(\hat{\partial}_{p,\lambda}\frac{\hat{S}}{2}\Big[\delta\hat{\mathfrak{h}}(x_p)\,,\,\hat{S}\Big\}_{\boldsymbol{L_{m_l}}}\Big)+$$
$$+\frac{\hat{S}}{2}\Big[\delta\hat{\mathfrak{h}}(x_p)\,,\,\hat{S}\Big\}_{\boldsymbol{I_{m_l}}}\frac{\hat{S}}{2}\Big[\delta\hat{\mathcal{K}}^{\boldsymbol{\mu}}(x_p)\,,\,\hat{S}\Big\}_{\boldsymbol{J_{m_l}}}\Big(\hat{\partial}_{p,\mu}\frac{\hat{S}}{2}\Big[\delta\hat{\mathfrak{h}}(x_p)\,,\,\hat{S}\Big\}_{\boldsymbol{K_{m_l}}}\frac{\hat{S}}{2}\Big[\delta\hat{\mathfrak{h}}(x_p)\,,\,\hat{S}\Big\}_{\boldsymbol{L_{m_l}}}\Big)+$$

$$
\begin{aligned}
+ \quad & \frac{\hat{S}}{2}\Big[\delta\hat{\mathfrak{h}}(x_p)\,,\hat{S}\Big\}_{\boldsymbol{I_{m_l}}}\frac{\hat{S}}{2}\Big[\delta\hat{\mathcal{K}}^{\boldsymbol{\mu}}(x_p)\,,\hat{S}\Big\}_{\boldsymbol{J_{m_l}}}\bigg(\hat{\partial}_{p,\mu}\frac{\hat{S}}{2}\Big[\delta\hat{\mathcal{K}}^{\boldsymbol{\lambda}}(x_p)\,,\hat{S}\Big\}_{\boldsymbol{K_{m_l}}}\Big(\hat{\partial}_{p,\lambda}\frac{\hat{S}}{2}\Big[\delta\hat{\mathfrak{h}}(x_p)\,,\hat{S}\Big\}_{\boldsymbol{L_{m_l}}}\Big)\bigg)+ \\
+ \quad & \frac{\hat{S}}{2}\Big[\delta\hat{\mathcal{K}}^{\boldsymbol{\nu}}(x_p)\,,\hat{S}\Big\}_{\boldsymbol{I_{m_l}}}\bigg(\hat{\partial}_{p,\nu}\frac{\hat{S}}{2}\Big[\delta\hat{\mathfrak{h}}(x_p)\,,\hat{S}\Big\}_{\boldsymbol{J_{m_l}}}\frac{\hat{S}}{2}\Big[\delta\hat{\mathfrak{h}}(x_p)\,,\hat{S}\Big\}_{\boldsymbol{K_{m_l}}}\frac{\hat{S}}{2}\Big[\delta\hat{\mathfrak{h}}(x_p)\,,\hat{S}\Big\}_{\boldsymbol{L_{m_l}}}\bigg)+ \\
+ \quad & \frac{\hat{S}}{2}\Big[\delta\hat{\mathcal{K}}^{\boldsymbol{\nu}}(x_p)\,,\hat{S}\Big\}_{\boldsymbol{I_{m_l}}}\bigg(\hat{\partial}_{p,\nu}\frac{\hat{S}}{2}\Big[\delta\hat{\mathfrak{h}}(x_p)\,,\hat{S}\Big\}_{\boldsymbol{J_{m_l}}}\frac{\hat{S}}{2}\Big[\delta\hat{\mathcal{K}}^{\boldsymbol{\lambda}}(x_p)\,,\hat{S}\Big\}_{\boldsymbol{K_{m_l}}}\Big(\hat{\partial}_{p,\lambda}\frac{\hat{S}}{2}\Big[\delta\hat{\mathfrak{h}}(x_p)\,,\hat{S}\Big\}_{\boldsymbol{L_{m_l}}}\Big)\bigg)+ \\
+ \quad & \frac{\hat{S}}{2}\Big[\delta\hat{\mathcal{K}}^{\boldsymbol{\nu}}(x_p)\,,\hat{S}\Big\}_{\boldsymbol{I_{m_l}}}\bigg(\hat{\partial}_{p,\nu}\frac{\hat{S}}{2}\Big[\delta\hat{\mathcal{K}}^{\boldsymbol{\mu}}(x_p)\,,\hat{S}\Big\}_{\boldsymbol{J_{m_l}}}\Big(\hat{\partial}_{p,\mu}\frac{\hat{S}}{2}\Big[\delta\hat{\mathfrak{h}}(x_p)\,,\hat{S}\Big\}_{\boldsymbol{K_{m_l}}}\frac{\hat{S}}{2}\Big[\delta\hat{\mathfrak{h}}(x_p)\,,\hat{S}\Big\}_{\boldsymbol{L_{m_l}}}\Big)\bigg)+ \\
+ \quad & \frac{\hat{S}}{2}\Big[\delta\hat{\mathcal{K}}^{\boldsymbol{\nu}}(x_p)\,,\hat{S}\Big\}_{\boldsymbol{I_{m_l}}}\bigg(\hat{\partial}_{p,\nu}\frac{\hat{S}}{2}\Big[\delta\hat{\mathcal{K}}^{\boldsymbol{\mu}}(x_p)\,,\hat{S}\Big\}_{\boldsymbol{J_{m_l}}}\Big(\hat{\partial}_{p,\mu}\frac{\hat{S}}{2}\Big[\delta\hat{\mathcal{K}}^{\boldsymbol{\lambda}}(x_p)\,,\hat{S}\Big\}_{\boldsymbol{K_{m_l}}}\Big(\hat{\partial}_{p,\lambda}\frac{\hat{S}}{2}\Big[\delta\hat{\mathfrak{h}}(x_p)\,,\hat{S}\Big\}_{\boldsymbol{L_{m_l}}}\Big)\Big)\bigg)\bigg]\bigg\rangle_{\hat{\boldsymbol{\mathcal{V}}}}\;;
\end{aligned}
$$

$$
\begin{aligned}
m &= 1;\; l = 1;\; \rightarrow\; C^{(n=4)}_{(2m=2),l=1} = 2;\; 'I_{m_l} = -';\; 'J_{m_l} = +';\; 'K_{m_l} = -';\; L_{m_l} = +'; \\
m &= 1;\; l = 2;\; \rightarrow\; C^{(n=4)}_{(2m=2),l=2} = 4;\; 'I_{m_l} = -';\; J_{m_l} = -';\; K_{m_l} = +';\; L_{m_l} = +'; \qquad \text{(D.68)} \\
m &= 2;\; l = 1;\; \rightarrow\; C^{(n=4)}_{(2m=4),(l=1)} = 1;\; 'I_{m_l} = -';\; 'J_{m_l} = -';\; K_{m_l} = -';\; 'L_{m_l} = -'\,.
\end{aligned}
$$

D.3 Lagrangian of the effective action with the bilinear, fermionic source fields

The gradient expansion of $\langle\mathcal{A}_{J_\psi}[\hat{T},\hat{\mathcal{V}};\hat{\mathcal{J}}]\rangle_{\hat{\mathcal{V}}}$ takes a considerably simpler form than that of $\langle\mathcal{A}_{DET}[\hat{T},\hat{\mathcal{V}};\hat{\mathcal{J}}]\rangle_{\hat{\mathcal{V}}}$ because it suffices to use for the propagation of anomalous doubled wavefunctions and fields, starting on the right-hand side with $J^a_{\psi;M}(x_p)$, the anomalous doubled Green function of the background potential $\langle\hat{\mathcal{V}}\rangle_{(3.59)}$ following from a saddle point approximation

$$
\begin{aligned}
\Big\langle\mathcal{A}_{J_\psi}[\hat{T},\hat{\mathcal{V}};\hat{\mathcal{J}}]\Big\rangle_{\hat{\boldsymbol{\mathcal{V}}}} &= \frac{1}{2}\int_C d^4x_p\;d^4y_q\;\times \qquad \text{(D.69)} \\
&\times J^{\dagger,b}_{\psi;N}(y_q)\;\hat{I}\bigg(\hat{T}(y_q)\;\Big\langle\hat{\mathcal{O}}^{\boldsymbol{-1};b'a'}_{N';M'}(y_q,x_p)\Big\rangle_{\hat{\boldsymbol{\mathcal{V}}}}\hat{T}^{-1}(x_p)-\hat{\mathcal{H}}^{\boldsymbol{-1};ba}_{N;M}(y_q,x_p)\bigg)\;\hat{I}\;J^a_{\psi;M}(x_p)\;; \\
\Big\langle\hat{\mathcal{O}}^{ba}_{N;M}(y_q,x_p)\Big\rangle_{\hat{\boldsymbol{\mathcal{V}}}} &= \bigg\{\langle\hat{\mathcal{H}}\rangle_{\hat{\mathcal{V}}}+\Big(\hat{T}^{-1}\langle\hat{\mathcal{H}}\rangle_{\hat{\mathcal{V}}}\hat{T}-\langle\hat{\mathcal{H}}\rangle_{\hat{\mathcal{V}}}\Big)+\hat{T}^{-1}\;\hat{I}\;\hat{S}\;\eta_q\frac{\hat{\mathcal{J}}^{b'a'}_{N';M'}(y_q,x_p)}{\mathcal{N}}\eta_p\;\hat{S}\;\hat{I}\;\hat{T}\bigg\}^{ba}_{N;M}(y_q,x_p). \qquad \text{(D.70)}
\end{aligned}
$$

Repeated application of the propagation rules (D.25-D.28) then determines the various coefficients with the background potential $\langle\hat{\mathcal{V}}(x_p)\rangle_{(3.59)}$. In analogy to the lowest order momentum expansion of $\langle\mathcal{A}_{DET}[\hat{T},\hat{\mathcal{V}};\hat{\mathcal{J}}]\rangle_{\hat{\mathcal{V}}}$ in section D.2, we neglect any resulting derivative terms of the fermionic source fields $(\hat{\partial}_{p,\kappa}J^a_{\psi;M}(x_p)\,)\approx 0$ from the action of 'unsaturated' gradients $\boldsymbol{\hat{\partial}_{p,\kappa}}$ and take into account the commutator for the action of these onto the background potentials within the Green functions

$$
\begin{aligned}
\Big\langle\mathcal{A}_{J_\psi}[\hat{T},\hat{\mathcal{V}};\hat{\mathcal{J}}\equiv 0]\Big\rangle_{\hat{\boldsymbol{\mathcal{V}}}} &= \frac{1}{2}\int_C d^4x_p\;d^4y_q\;J^{\dagger,b}_{\psi;N}(y_q)\;\hat{I}\bigg(\hat{T}(y_q)\;\times \qquad \text{(D.71)} \\
\times\;&\bigg(\sum_{n=1}^{\infty}(-1)^n\Big[\langle\hat{\mathcal{H}}\rangle^{\boldsymbol{-1}}_{\hat{\mathcal{V}}}\Big(\Delta\langle\hat{\mathcal{H}}\rangle_{\hat{\mathcal{V}}}\;\;\langle\hat{\mathcal{H}}\rangle^{\boldsymbol{-1}}_{\hat{\mathcal{V}}}\Big)^n\Big]^{b'a'}_{N';M'}(y_q,x_p)\bigg)\;\;\hat{T}^{-1}(x_p)\bigg)^{ba}_{N;M}\hat{I}\;J^a_{\psi;M}(x_p) \\
=\;&\frac{1}{2}\sum_{n=1}^{\infty}(-1)^n\int_C d^4x_p\;d^4y_q\;J^{\dagger,b}_{\psi;N}(y_q)\;\hat{I}\bigg(\hat{T}(y_q)\;\times \\
\times\;&\Big[\langle\hat{\mathcal{H}}\rangle^{\boldsymbol{-1}}_{\hat{\mathcal{V}}}\Big(\delta\hat{\mathfrak{h}}\;\;\langle\hat{\mathcal{H}}\rangle^{\boldsymbol{-1}}_{\hat{\mathcal{V}}}+\delta\hat{\mathcal{K}}^{\boldsymbol{\kappa}}\boldsymbol{\hat{\partial}_{p,\kappa}}\;\;\langle\hat{\mathcal{H}}\rangle^{\boldsymbol{-1}}_{\hat{\mathcal{V}}}\Big)^n\Big]^{b'a'}_{N';M'}(y_q,x_p)\;\hat{T}^{-1}(x_p)\bigg)^{ba}_{N;M}\hat{I}\;J^a_{\psi;M}(x_p)\;; \\
&\big(\hat{\partial}_{p,\kappa}J^a_{\psi;M}(x_p)\,\big)\approx 0\;;\quad\Big[\boldsymbol{\hat{\partial}_{p,\kappa}}\;,\langle\hat{\mathcal{H}}\rangle^{\boldsymbol{-1}}_{\hat{\mathcal{V}}}\Big]_-\simeq -\,\langle\hat{\mathcal{H}}\rangle^{\boldsymbol{-1}}_{\hat{\mathcal{V}}}\;\underbrace{\big(\hat{\partial}_{p,\boldsymbol{\kappa}}\mathcal{V}_{\alpha;\kappa_1}\big)\;\hat{V}^{\kappa_1}_{\alpha}}_{(\,\hat{\partial}_{p,\boldsymbol{\kappa}}\hat{\mathfrak{V}}\,)}\;\langle\hat{\mathcal{H}}\rangle^{\boldsymbol{-1}}_{\hat{\mathcal{V}}}\;. \qquad \text{(D.72)}
\end{aligned}
$$

Since the action $\langle\mathcal{A}_{J_\psi}[\hat{T},\hat{\mathcal{V}};\hat{\mathfrak{J}}]\rangle_{\hat{\mathcal{V}}}$ does not contain a back-propagation to the same spacetime point as $\langle\mathcal{A}_{DET}[\hat{T},\hat{\mathcal{V}};\hat{\mathfrak{J}}]\rangle_{\hat{\mathcal{V}}}$, due to the missing of corresponding traces, any combinatorial factors cannot occur in the expansion of $\langle\mathcal{A}_{J_\psi}[\hat{T},\hat{\mathcal{V}};\hat{\mathfrak{J}}]\rangle_{\hat{\mathcal{V}}}$. Therefore, we can simply list the final approximated gradient expansion of $\langle\mathcal{A}_{J_\psi}[\hat{T},\hat{\mathcal{V}};\hat{\mathfrak{J}}]\rangle_{\hat{\mathcal{V}}}$ in relation (D.73) which can be further reduced by the assumption of constant background potentials $(\hat{\partial}_{p,\kappa}\mathcal{V}_{\alpha;\mu}(x_p)\,)\approx 0$

$$\begin{aligned}
&\Big\langle\mathcal{A}_{J_\psi}[\hat{T},\hat{\mathcal{V}};\hat{\mathfrak{J}}\equiv 0]\Big\rangle_{\hat{\boldsymbol{\mathcal{V}}}}=\frac{1}{2}\sum_{n=1}^{\infty}(-1)^n\int_C d^4x_p\;d^4y_q\;J_{\psi;N}^{\dagger,b}(y_q)\;\hat{I}\;\hat{T}_{N;N'}^{bb'}(y_q)\times \qquad\text{(D.73)}\\
&\times\Bigg[\bigg[\langle\hat{\mathfrak{H}}\rangle_{\hat{\mathcal{V}}}^{\boldsymbol{-1}}\bigg(\Big(\delta\hat{\mathfrak{h}}-\delta\hat{\mathfrak{K}}^{\boldsymbol{\kappa_n}}\;\langle\hat{\mathfrak{H}}\rangle_{\hat{\mathcal{V}}}^{\boldsymbol{-1}}\;\big(\hat{\partial}_{p,\boldsymbol{\kappa_n}}\hat{\mathfrak{V}}\big)\Big)\,\langle\hat{\mathfrak{H}}\rangle_{\hat{\mathcal{V}}}^{\boldsymbol{-1}}+\delta\hat{\mathfrak{K}}^{\boldsymbol{\kappa_n}}\;\langle\hat{\mathfrak{H}}\rangle_{\hat{\mathcal{V}}}^{\boldsymbol{-1}}\;\hat{\boldsymbol{\partial}}_{\boldsymbol{p,\kappa_n}}\bigg)\bigg]\times\\
&\times\bigg[\langle\hat{\mathfrak{H}}\rangle_{\hat{\mathcal{V}}}^{\boldsymbol{-1}}\bigg(\Big(\delta\hat{\mathfrak{h}}-\delta\hat{\mathfrak{K}}^{\boldsymbol{\kappa_{n-1}}}\;\langle\hat{\mathfrak{H}}\rangle_{\hat{\mathcal{V}}}^{\boldsymbol{-1}}\;\big(\hat{\partial}_{p,\boldsymbol{\kappa_{n-1}}}\hat{\mathfrak{V}}\big)\Big)\,\langle\hat{\mathfrak{H}}\rangle_{\hat{\mathcal{V}}}^{\boldsymbol{-1}}+\delta\hat{\mathfrak{K}}^{\boldsymbol{\kappa_{n-1}}}\;\langle\hat{\mathfrak{H}}\rangle_{\hat{\mathcal{V}}}^{\boldsymbol{-1}}\;\hat{\boldsymbol{\partial}}_{\boldsymbol{p,\kappa_{n-1}}}\bigg)\bigg]\times\\
&\times\;\vdots\\
&\times\bigg[\langle\hat{\mathfrak{H}}\rangle_{\hat{\mathcal{V}}}^{\boldsymbol{-1}}\bigg(\Big(\delta\hat{\mathfrak{h}}-\delta\hat{\mathfrak{K}}^{\boldsymbol{\kappa_2}}\;\langle\hat{\mathfrak{H}}\rangle_{\hat{\mathcal{V}}}^{\boldsymbol{-1}}\;\big(\hat{\partial}_{p,\boldsymbol{\kappa_2}}\hat{\mathfrak{V}}\big)\Big)\,\langle\hat{\mathfrak{H}}\rangle_{\hat{\mathcal{V}}}^{\boldsymbol{-1}}+\delta\hat{\mathfrak{K}}^{\boldsymbol{\kappa_2}}\;\langle\hat{\mathfrak{H}}\rangle_{\hat{\mathcal{V}}}^{\boldsymbol{-1}}\;\hat{\boldsymbol{\partial}}_{\boldsymbol{p,\kappa_2}}\bigg)\bigg]\times\\
&\times\bigg[\langle\hat{\mathfrak{H}}\rangle_{\hat{\mathcal{V}}}^{\boldsymbol{-1}}\bigg(\Big(\delta\hat{\mathfrak{h}}-\delta\hat{\mathfrak{K}}^{\boldsymbol{\kappa_1}}\;\langle\hat{\mathfrak{H}}\rangle_{\hat{\mathcal{V}}}^{\boldsymbol{-1}}\;\big(\hat{\partial}_{p,\boldsymbol{\kappa_1}}\hat{\mathfrak{V}}\big)\Big)\,\langle\hat{\mathfrak{H}}\rangle_{\hat{\mathcal{V}}}^{\boldsymbol{-1}}+\delta\hat{\mathfrak{K}}^{\boldsymbol{\kappa_1}}\;\langle\hat{\mathfrak{H}}\rangle_{\hat{\mathcal{V}}}^{\boldsymbol{-1}}\;\hat{\boldsymbol{\partial}}_{\boldsymbol{p,\kappa_1}}\bigg)\bigg]\Bigg]_{\boldsymbol{N';M'}}^{\boldsymbol{b'a'}}\!\!\boldsymbol{(y_q,x_p)}\;\hat{T}_{M'M}^{-1;a'a}(x_p)\;\hat{I}\;J_{\psi;M}^{a}(x_p)\,.
\end{aligned}$$

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